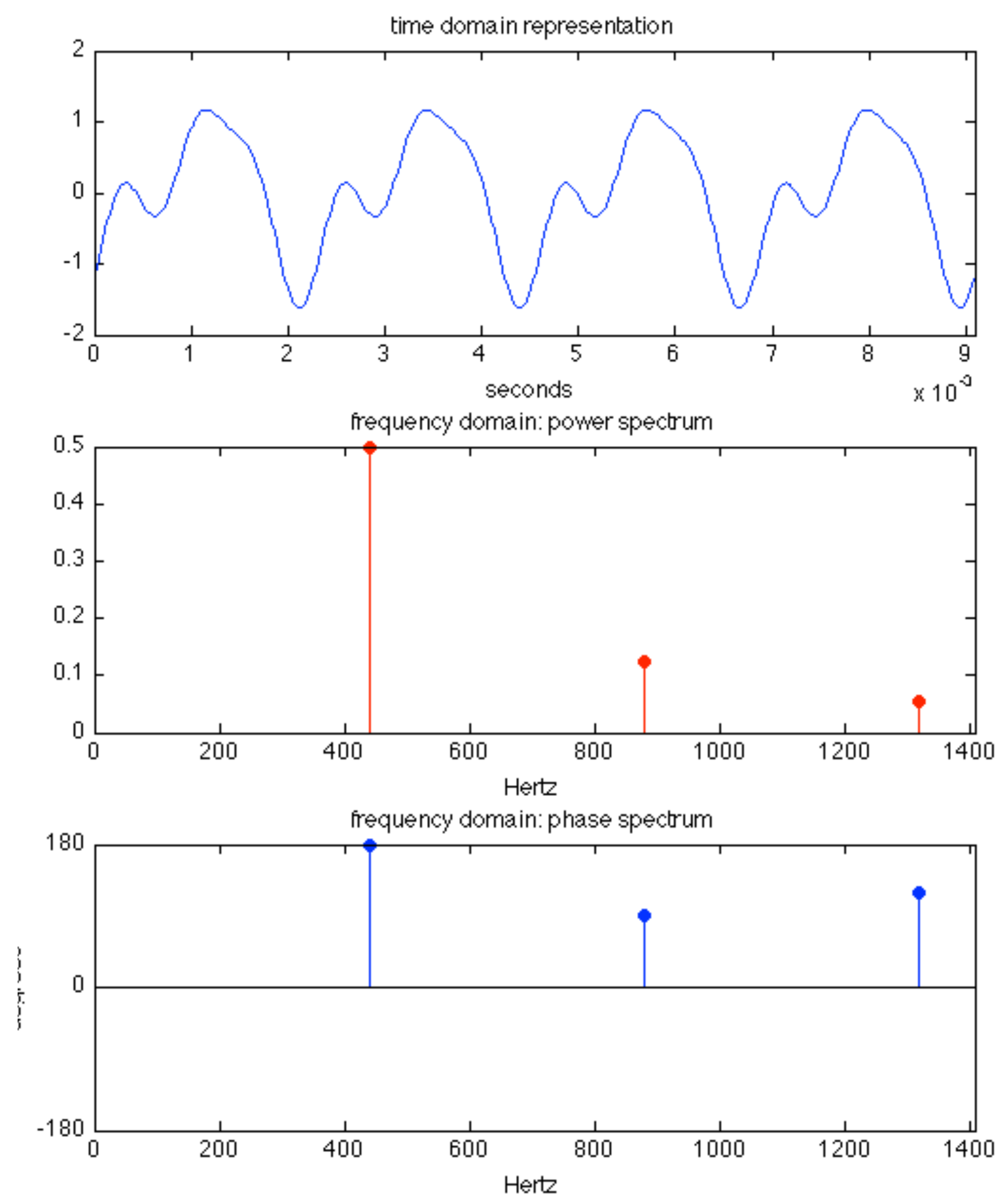
Math 345: Applied Mathematics

Introduction to Fourier Series, II

Time Domain, Frequency Domain



Marcus Pendergrass

Hampden-Sydney College

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Fourier's Theorem

- If s in V, and $f_0 = 1/T$, then

$$s(t) = \sum_{k=1}^{\infty} a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t)$$

where

$$a_k = \frac{2}{T} \int_0^T s(t)\, \cos(2\pi k f_0 t)\, dt\,, \quad b_k = \frac{2}{T} \int_0^T s(t)\, \sin(2\pi k f_0 t)\, dt$$

Parseval's Relation

- If s in V has Fourier series

$$s(t) = \sum_{k=1}^{\infty} a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t)$$

then the total energy in s is

$$\|s\|^2 = \frac{T}{2} \sum_{k=1}^{\infty} a_k^2 + b_k^2$$

Components

- Definition. The *component of s at frequency kf_0* is

$$s_k(t) = a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t)$$

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So

$$s(t) = \sum_{k=1}^{\infty} s_k(t)$$

Components

- Note: The component of s at frequency kf_0 can be written as

$$\begin{aligned} s_k(t) &= a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t) \\ &= A_k \cos(2\pi k f_0 t - \phi_k) \end{aligned}$$

where

$$\|s_k\|^2 = \frac{T}{2} A_k^2 = \frac{T}{2}\left(a_k^2 + b_k^2\right)$$

$$\tan(\phi_k) = \frac{b_k}{a_k}$$

Energy and Phase Components

- **Definition**: The *energy of* s *at frequency* kf_0 is

$$\|s_k\|^2 = \frac{T}{2}A_k^2 = \frac{T}{2}\left(a_k^2 + b_k^2\right)$$

and the *phase of* s *at* kf_0 is the angle ϕ_k satisfying

$$\tan(\phi_k) = \frac{b_k}{a_k}, \quad -\pi < \phi_k \leq \pi$$

Power

- Power is energy per unit time.

Energy on $[0, T]$

$$\int_0^T s(t)^2 \, dt = \|s\|^2$$

Average power on $[0, T]$

$$\frac{1}{T}\int_0^T s(t)^2 \, dt = \frac{\|s\|^2}{T}$$

Power Components

- Definition: The *power of* s *at frequency* kf_0 is

$$\frac{\|s_k\|^2}{T} = \frac{1}{2}A_k^2 = \frac{1}{2}\left(a_k^2 + b_k^2\right)$$

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Note: this is independent of T

Energy Spectrum

- **Definition**: The *energy spectrum of* s is the set of pairs

$$\left(kf_0, \frac{T}{2}(a_k^2 + b_k^2)\right), \quad k = 1, 2, 3, \ldots$$

Power Spectrum

- **Definition**: The *power spectrum of* s is the set of pairs

$$\left(kf_0, \frac{1}{2}(a_k^2 + b_k^2)\right) : \quad k = 1, 2, 3, \ldots$$

Phase Spectrum

- **Definition**: The *phase spectrum of* s consists of the pairs

$$\left(kf_0, \phi_k\right), \quad k = 1, 2, 3, \ldots$$

Example 1

- Find the power and phase spectra of

$$s(t) = -\cos(880\pi t) + \frac{1}{2}\sin(1760\pi t) - \frac{1}{6}\cos(2640\pi t) + \frac{\sqrt{3}}{6}\sin(2640\pi t)$$

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solution

- $f_0 = 440$
- $n = 3$
- $a = (-1, 0, -1/6)$
- $b = (0, 1/2, \sqrt{3}/6)$

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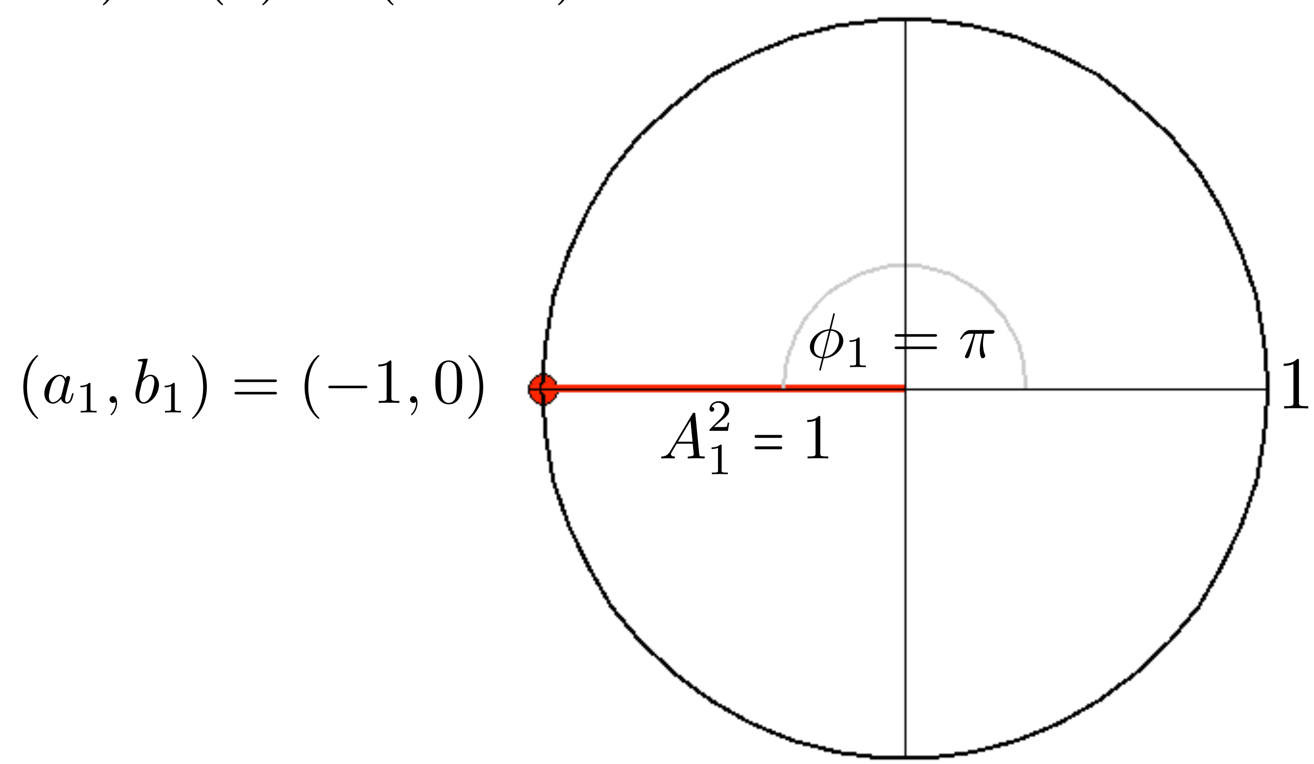
<u>solution</u>

$$s_1(t) = (-1)\cos(880\pi t) + (0)\sin(880\pi t)$$

- $f_0 = 440$
- $n = 3$
- $a = (-1, 0, -1/6)$
- $b = (0, 1/2, \sqrt{3}/6)$



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<u>solution</u>

$$s_2(t) = (0)\cos(1760\pi t) + \left(\frac{1}{2}\right)\sin(1760\pi t)$$

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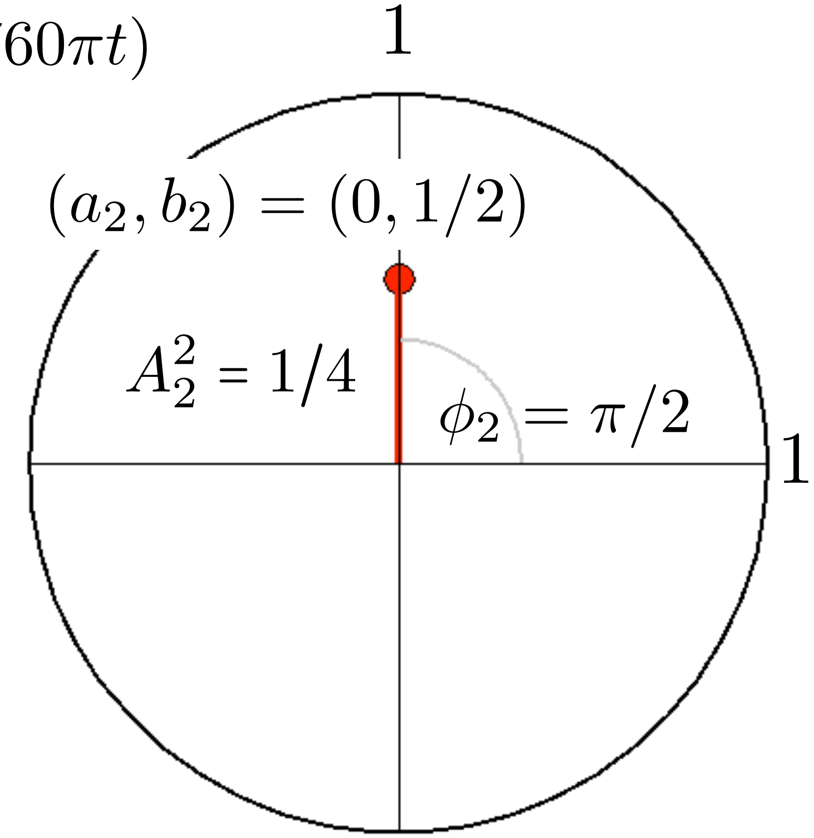

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<u>solution</u>

$$s_3(t) = \left(-\frac{1}{6}\right)\cos(2640\pi t) + \left(\frac{\sqrt{3}}{6}\right)\sin(2640\pi t)$$

- $f_0 = 440$
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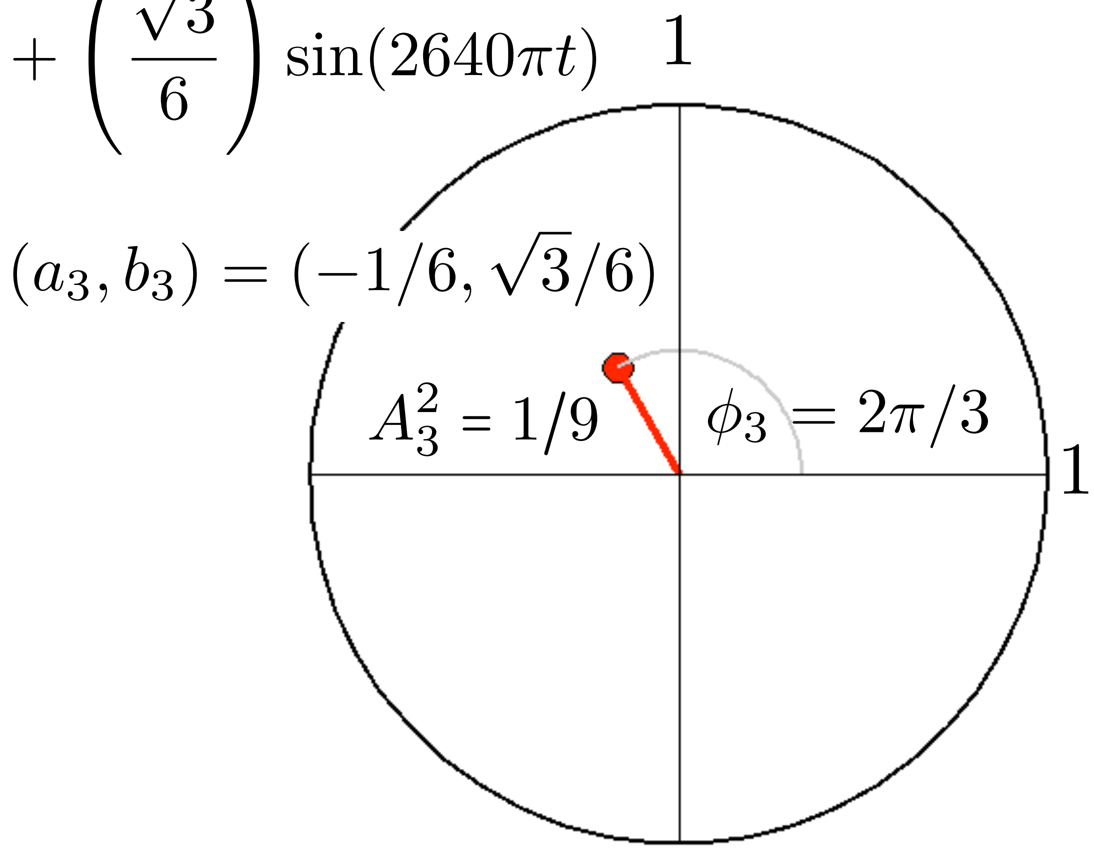

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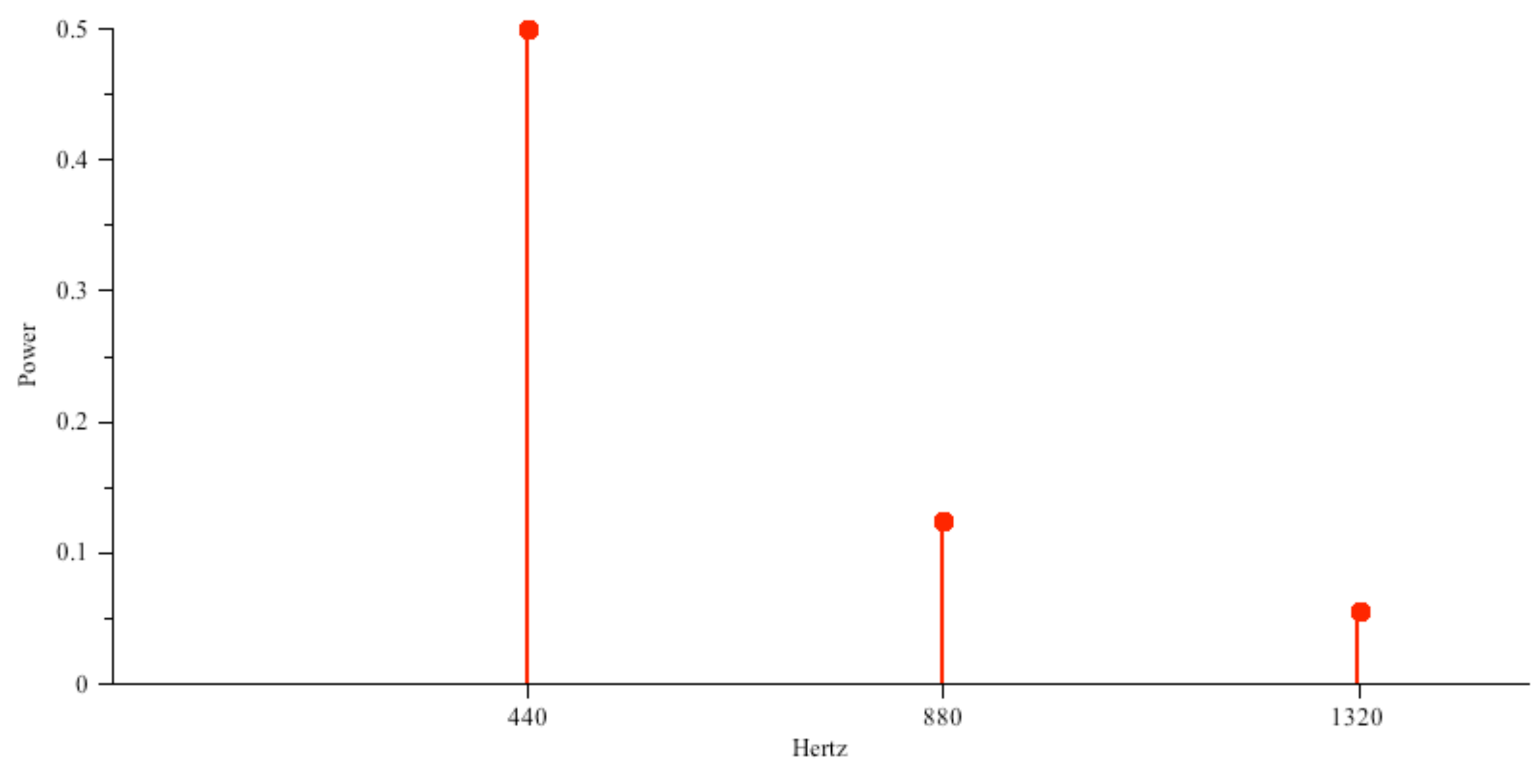
<u>solution</u>

Power Spectrum

Frequency	Power
440	1/2
880	1/8
1320	1/18



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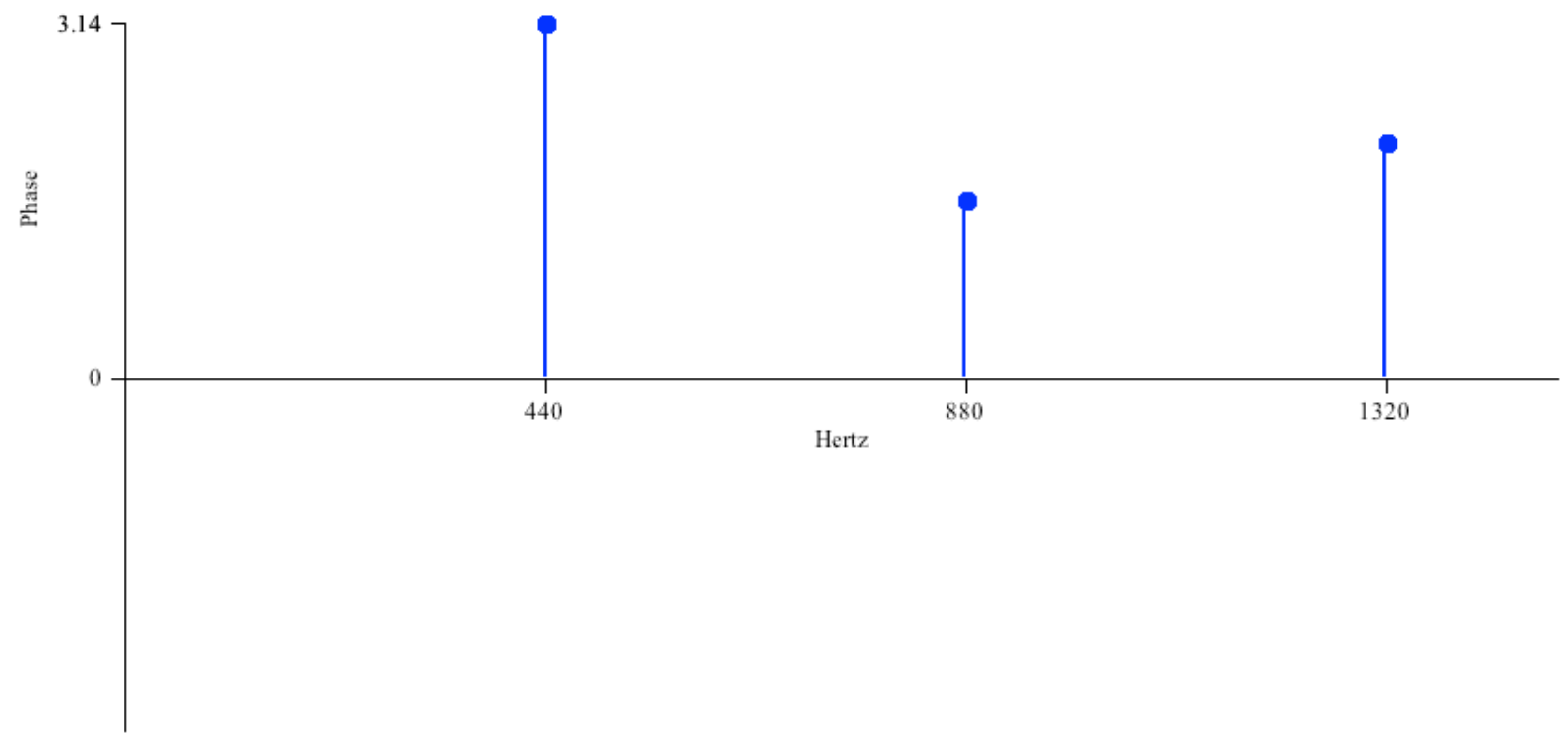
<u>solution</u>

Phase Spectrum

Frequency	Phase
440	π
880	$\pi/2$
1320	$2\pi/3$



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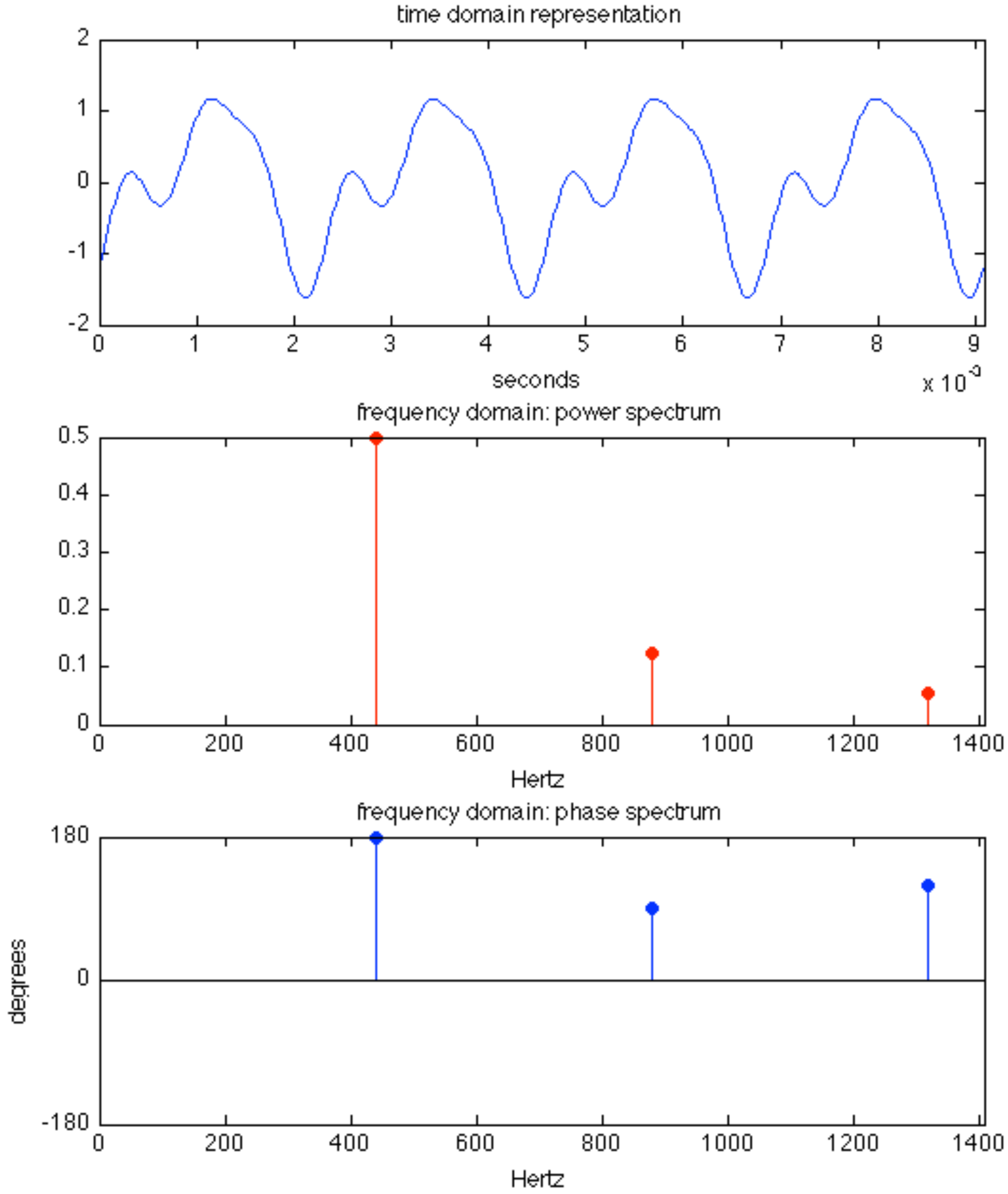
<u>solution</u>

Note that we can re-write s now:

$$s(t) = \cos(880\pi t - \pi) + \frac{1}{2}\cos(1760\pi t - \pi/2) + \frac{1}{3}\cos(2640\pi t - 2\pi/3)$$

Time Domain, Frequency Domain



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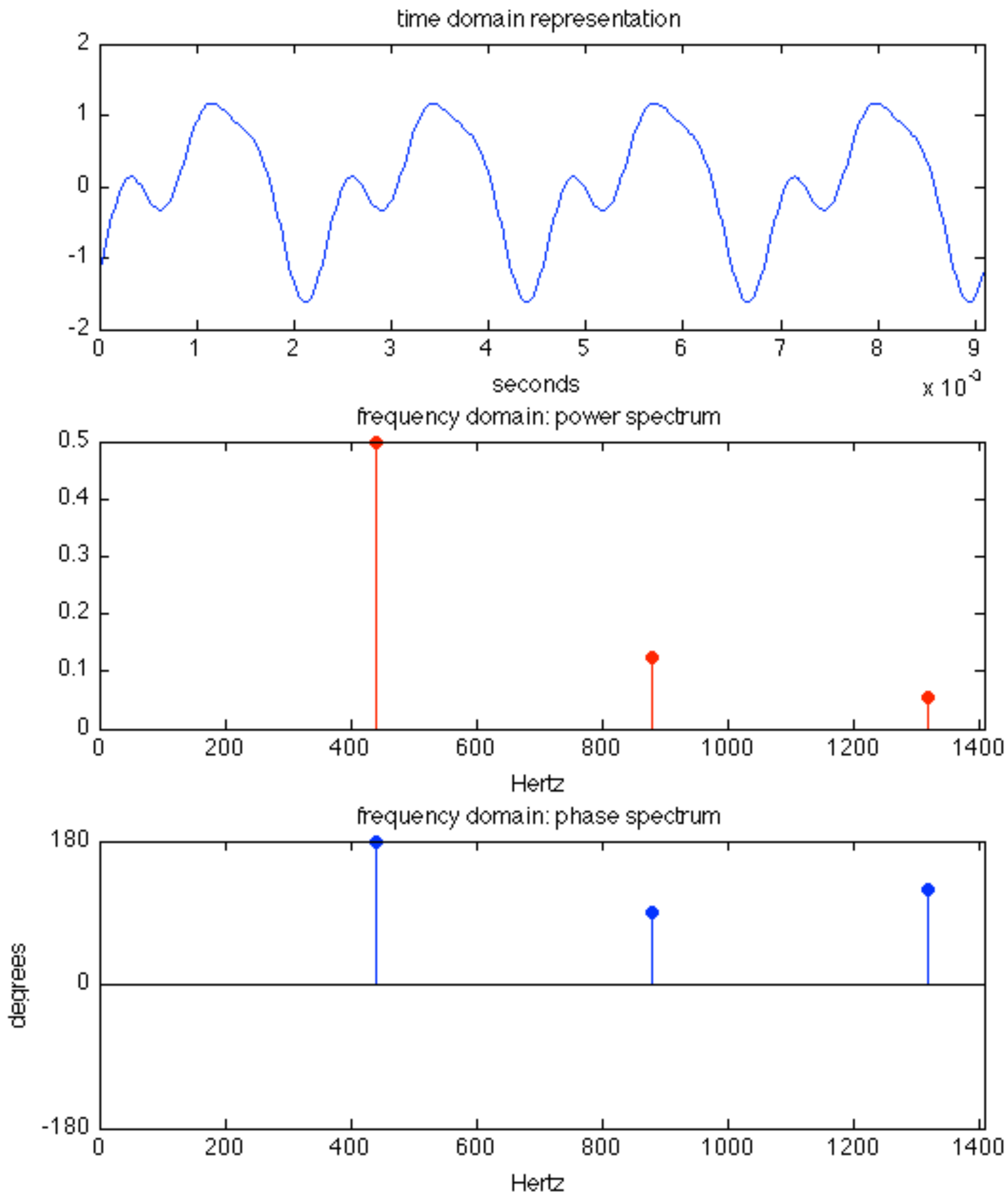
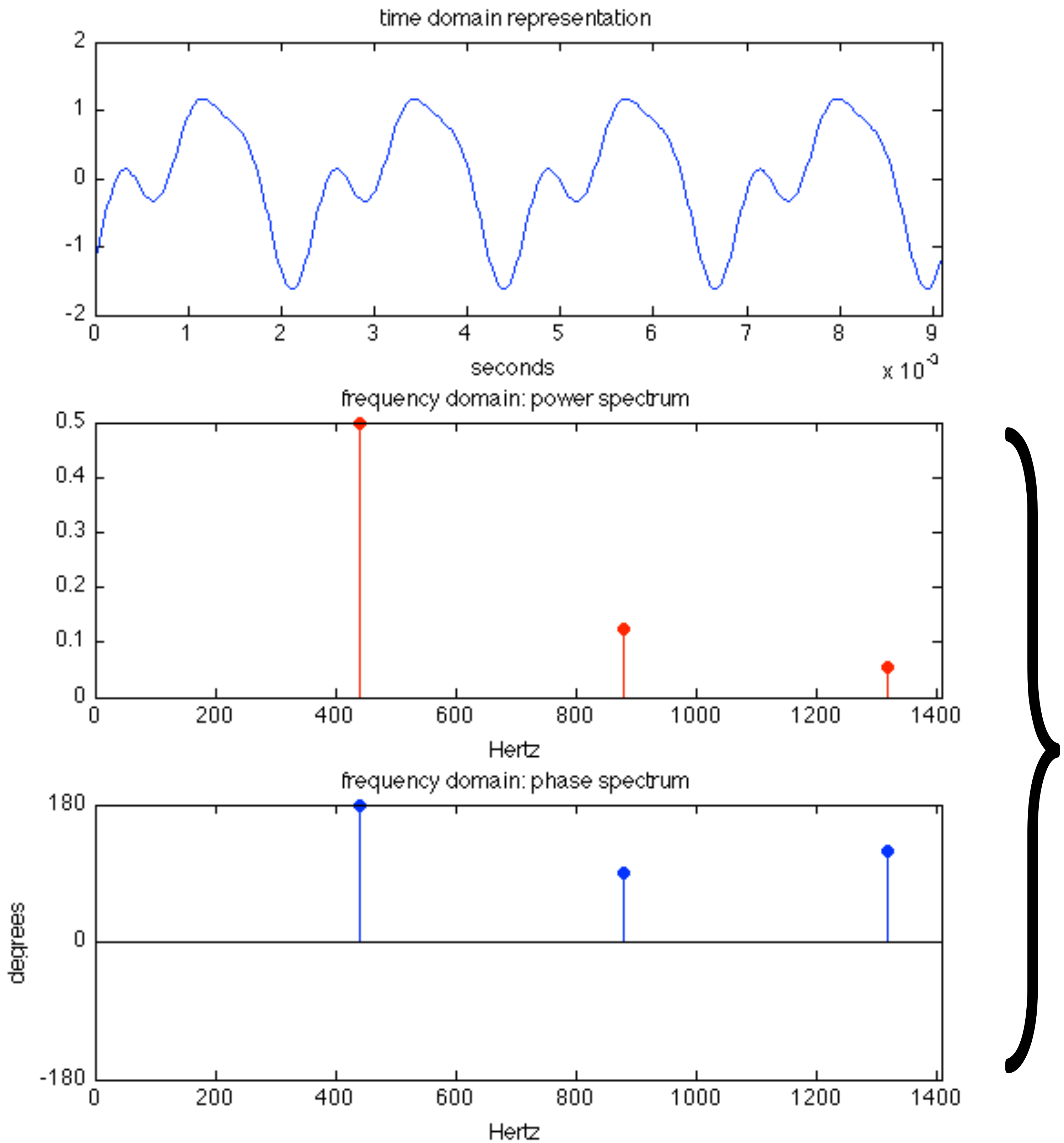


Time domain representation

Time Domain, Frequency Domain

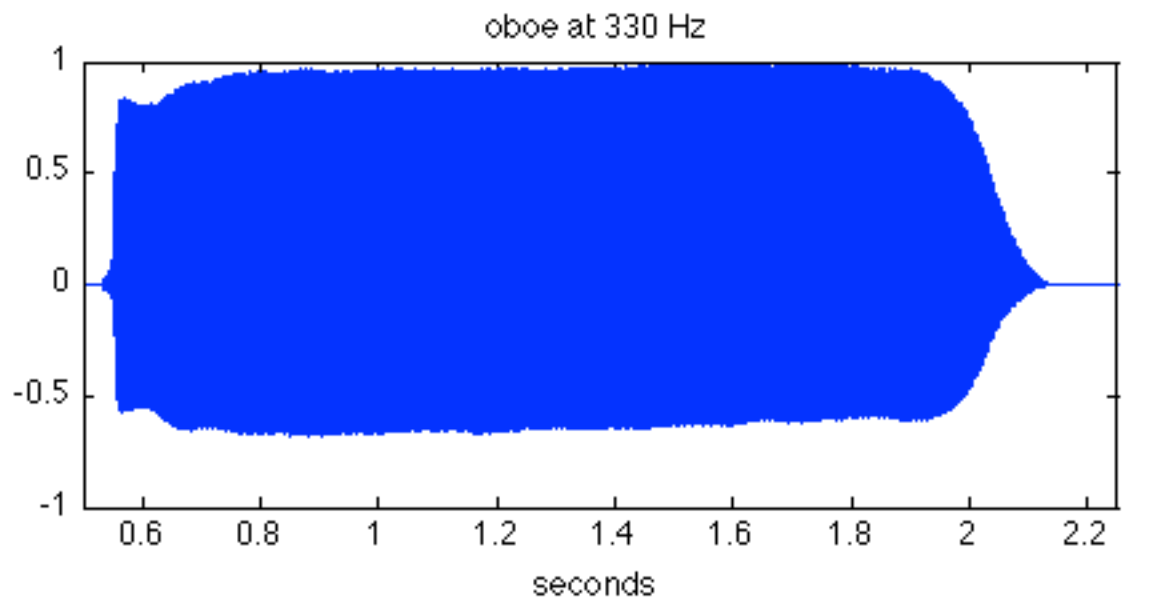


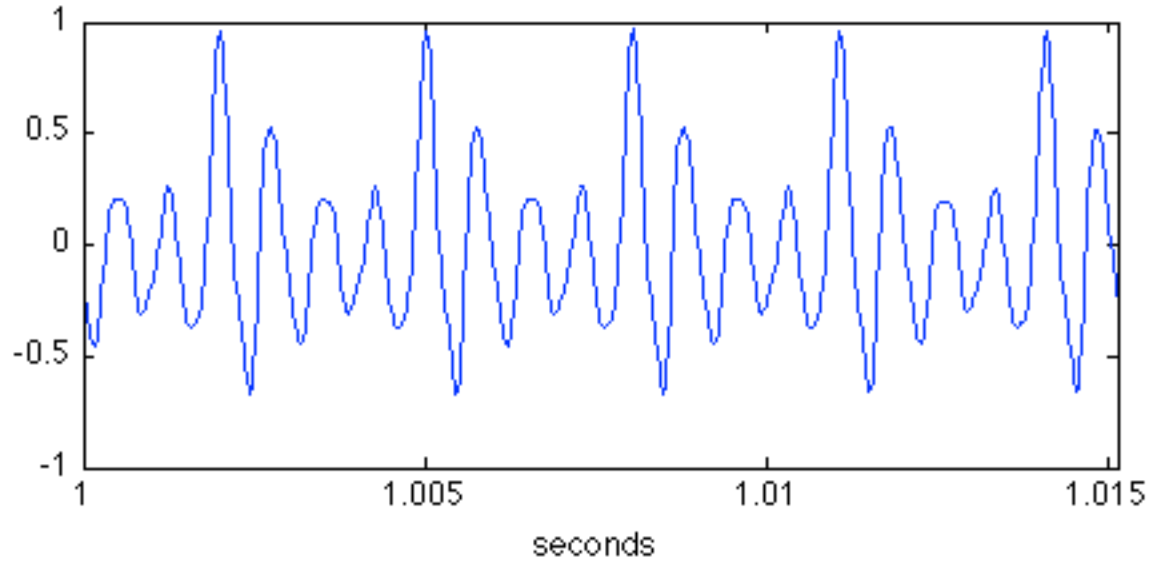
Time domain representation

Frequency domain representation

Real Signals

- At right is the time-domain waveform for an oboe, playing at 330 Hz (middle E).
- Let's try to approximate this sound with a Fourier projection.
- Expect frequency content at 330, 660, 990, 1320, etc. (More on this later.)





Real Signals

- Approximate the Fourier coefficients for these frequencies numerically:

$$a_k = \frac{2}{T}\int_0^T s(t)\cos(2\pi k 330 t)\,dt$$

$$\approx \frac{2}{T}\sum_{j=1}^{N} s(t_j)\cos(2\pi k 330 t_j)\,\Delta t$$

$$b_k = \frac{2}{T}\int_0^T s(t)\sin(2\pi k 330 t)\,dt$$

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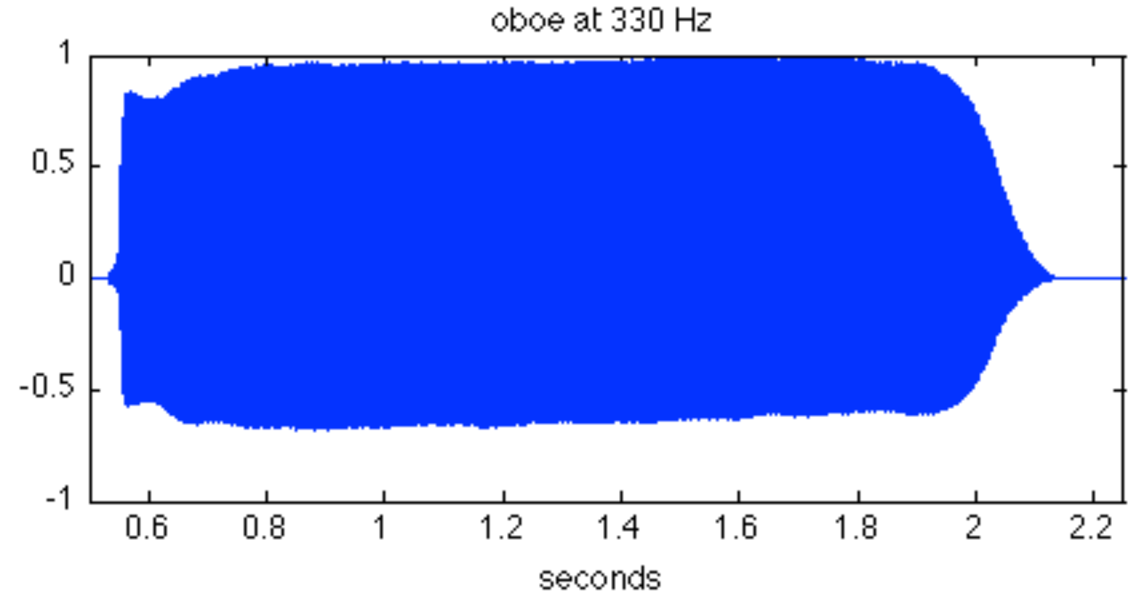

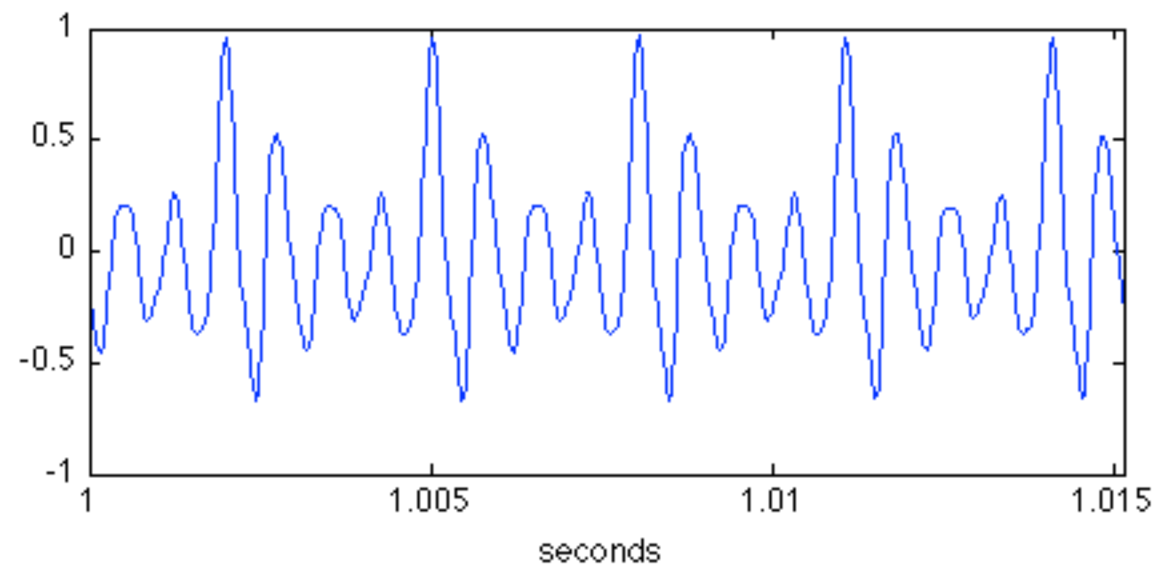

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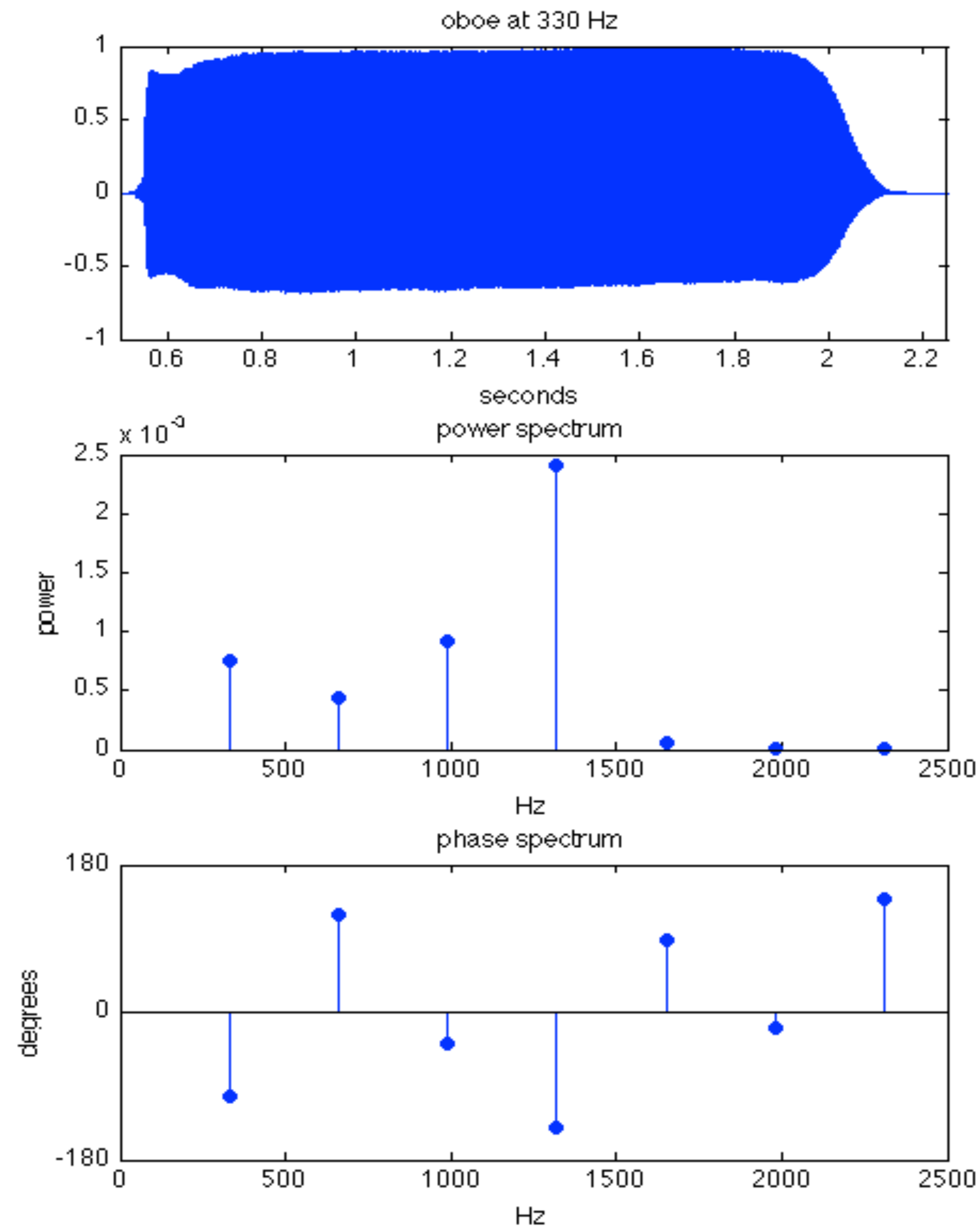
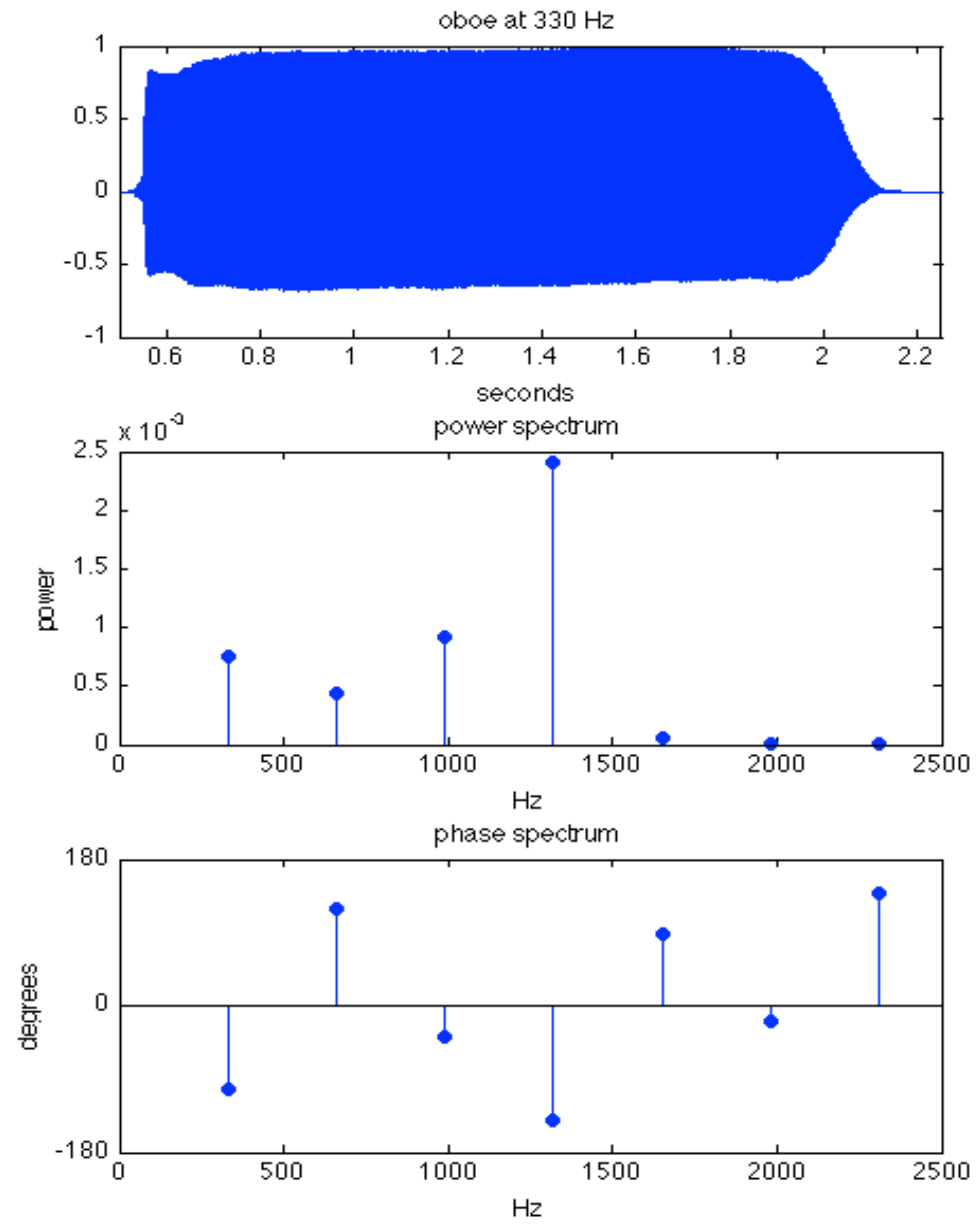


Real Signals

- Now use the Fourier coefficients to reconstruct the signal:

$$\hat{s}(t) = \sum_{k=1}^{7} a_k \cos(2\pi 330 kt) + b_k \sin(2\pi 330 kt)$$

- Compare the signal and the approximation...



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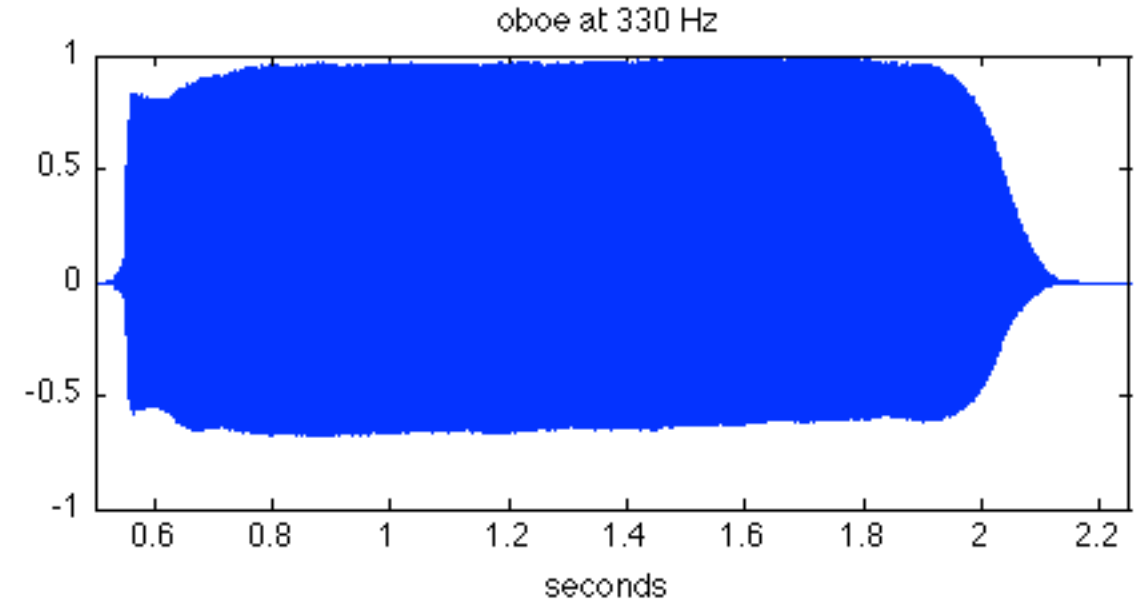

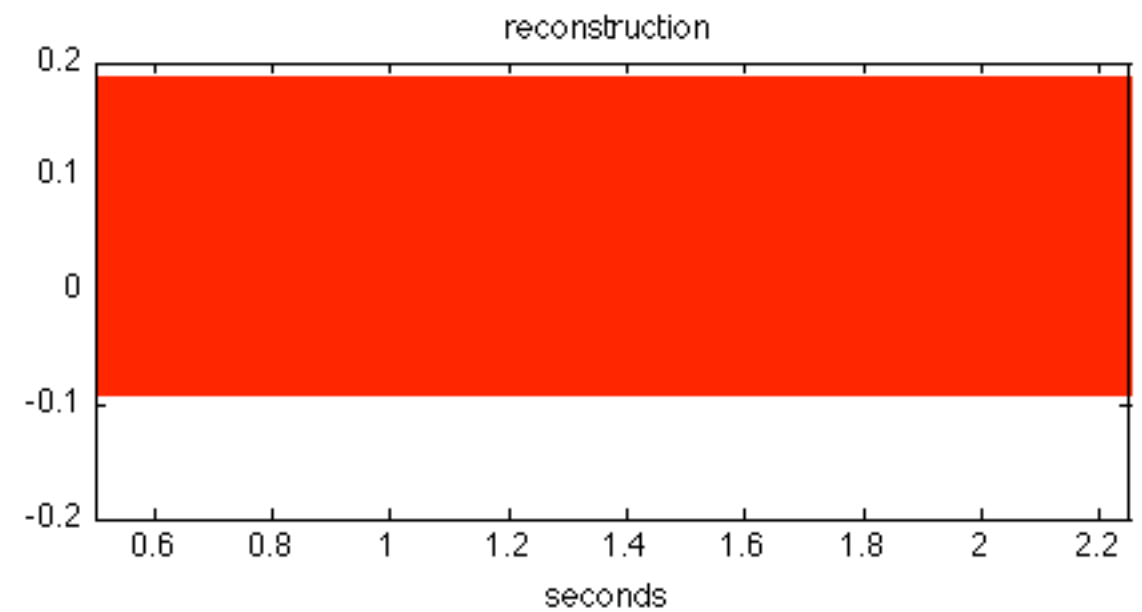

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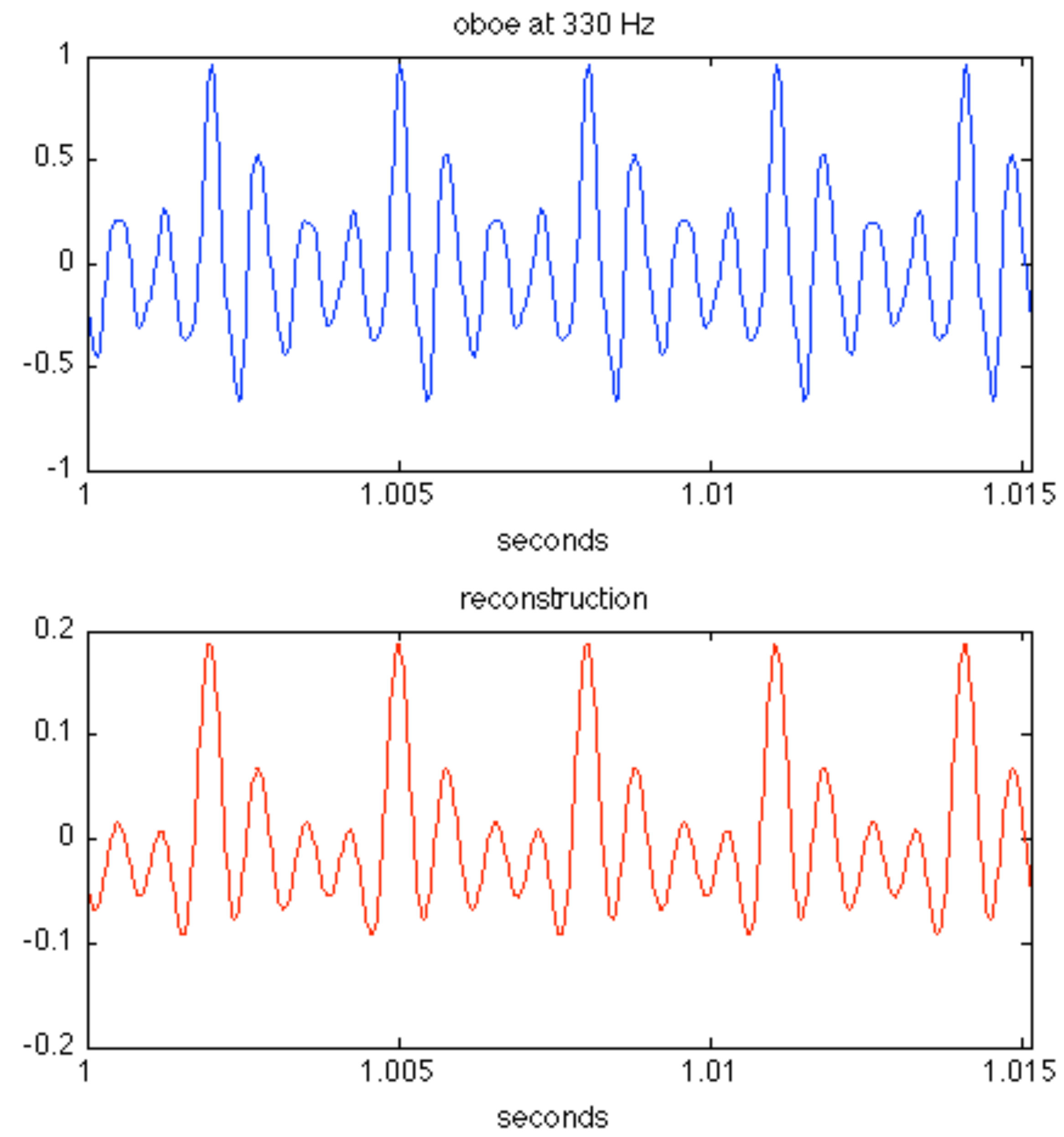
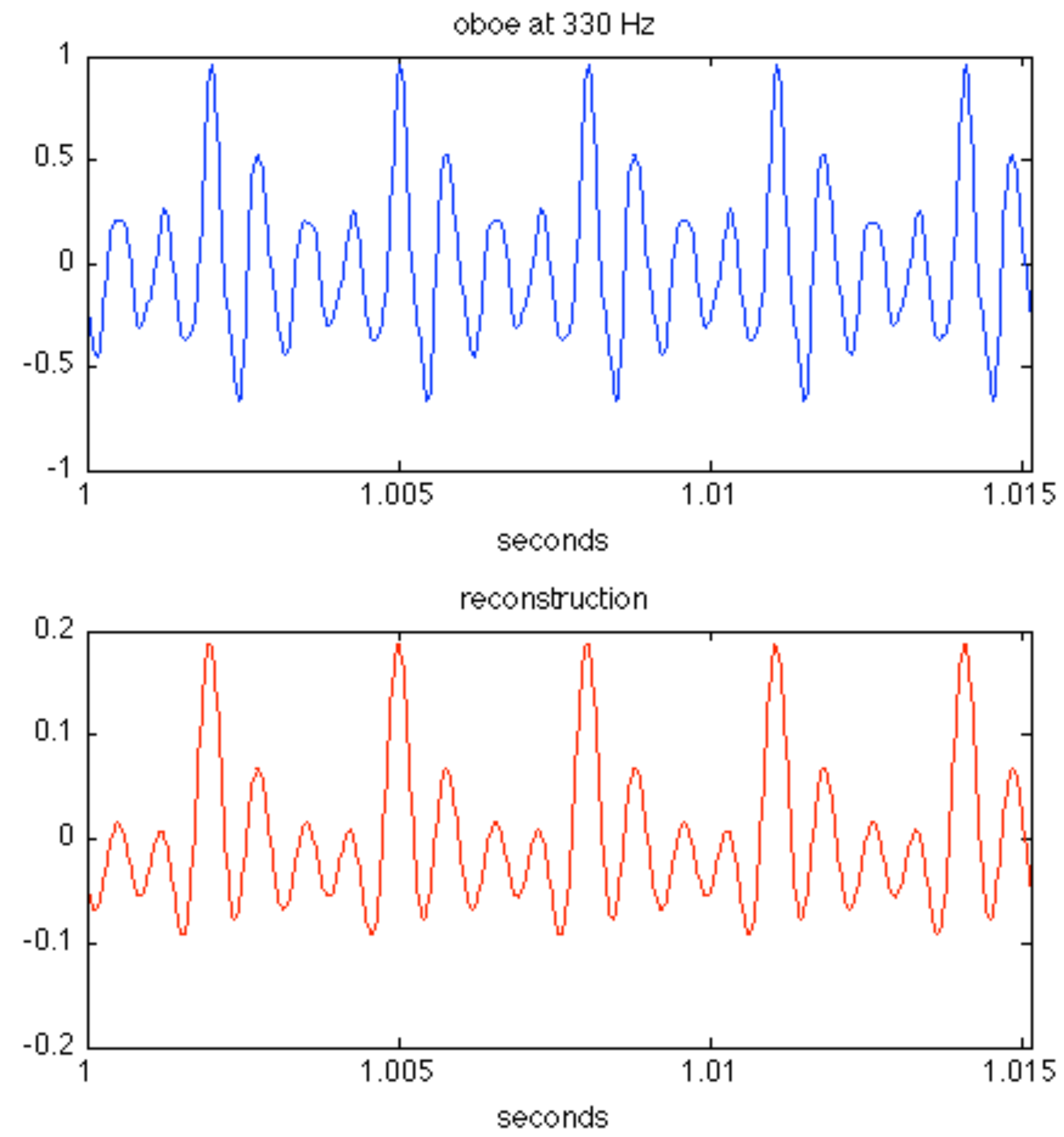


Real Signals

- Notes
  - Good match of waveform shape at the micro-level (but some energy is missing)



Real Signals

- <u>Note</u>:
  - Good match of waveform shape at the micro-level (but some energy is missing)
  - Macro-level details are not being modeled
  - What is getting left out?