On my honor, I have neither given nor received any aid on this work, nor am I aware of any breach of the Honor Code that I shall not immediately report.

Pledged: _____

Print Name: _____

1. A pool of twenty workers are available to perform ten different jobs. (Each job requires just one worker.) How many assignments of workers to jobs are possible?

Solution:

Fix an ordering for the ten jobs, then assign workers to the jobs. There are 20 choices for the worker for job 1, then 19 remaining choices for the worker for job 2, and so on. So there are

$$P_{20,10} = \frac{20!}{10!} = 670,442,572,800$$

possible assignments in total.

2. Several states, including Virginia and New York, issue license plate with the format ABC-1234 – in other words the first three characters are capital letters, and the last four are digits.

   (a) How many license plates are there with this format?

   (b) Unlike Virginia, New York has reserved plate numbers AAA-0000 through FAA-0999 for state use. How many such reserved plates are there?

Solution:

   (a) One can first choose the three letters, then the four numbers. Since repeats are allowed, we have

   $$26^3 \cdot 10^4 = 175,760,000$$

   possibilities in all.

   (b) Each three-letter sequence from AAA to EZZ is associated with 10,000 reserved plates (numbered 0000 through 9999). There are $5 \cdot 26 \cdot 26$ such three-letter sequences. In addition, the sequence FAA is associated with 1,000 reserved plates (numbered 0000 through 0999). So the total number of reserved plates is
   $$5 \cdot 26^2 \cdot 10^4 + 1 \cdot 10^3 = 33,801,000$$

3. If 4 Americans, 3 Frenchmen, and 5 Britons are to be seated in a row, how many seating arrangements are possible if people of the same nationality must sit next to each other?

Solution:

First order the individuals in each nationality, then order the nationalities. There are

$$4! \cdot 3! \cdot 5! \cdot 3! = 103,680$$

possible orderings.

4. Find an *elementary function* $f(x)$ that grows faster than $n!$. (An elementary function is a finite combination of polynomials, trig functions, and exponential/logarithmic functions. Basically, these are the kinds of

functions that are seen in a first course in calculus.) If you can, make sure your function grows <u>much</u> faster than $n!$, in the following sense:

$$\lim_{n \to \infty} \frac{n!}{f(n)} = 0$$

Solution:

One such function is $f(x) = x^x$. Then we have

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n}$$
$$\leq \frac{1}{n}$$

Hence $\lim_{n \to \infty} n!/n^n = 0$ by the Squeezing Theorem.

5. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many such choices are there if

   (a) there are no restrictions?
   (b) A and B will not serve together?
   (c) C and D will serve together or not at all?
   (d) E must be an officer?
   (e) F will serve only if he is president?

Solution:

   (a) If there are no restrictions, this is a permutation of 10 things taken 3 at a time. There are $P_{10,3} = 720$ such permutations.

   (b) There are three separate cases to consider: neither A nor B serves, A serves but B does not, and B serves but A does not. If neither A nor B is to serve, then we simply pick a permutation of 3 of the remaining 8 club members for the officer slate. If A serves but not B, we first choose the job to give A, then choose a permutation of 2 of the remaining 8 club members for the other two positions. The same logic is in force when B serves but not A. Hence, the total number of possibilities is $P_{8,3} + 3 \cdot P_{8,2} + 3 \cdot P_{8,2} = 672$.

   (c) This is similar to the previous problem, except that there are two cases rather than three. In the first, both C and D serve, and there are $3 \cdot 2 \cdot 8$ possibilities here. In the second, neither C not D serves, and there are $P_{8,3}$ possibilities here. So there are $3 \cdot 2 \cdot 8 + P_{8,3} = 384$ possibilities in all.

   (d) Choose the position for E, then order 2 of the remaining 9 people for the other two positions. Thus, there are $3 \cdot P_{9,2} = 216$ possibilities.

   (e) There are again two cases: F serves as president, or F does not serve at all. In the first case, there are $1 \cdot P_{9,2}$ possibilities, while in the second there are $P_{9,3}$. So the total number of possibilities is $1 \cdot P_{9,2} + P_{9,3} = 576$.

6. A campus club has 4 freshmen, 5 sophomores, 3 juniors, and 6 seniors. The club wants to choose a delegation of size 4 to speak to the college president. How many possible selections are there if

   (a) there are no restrictions?
   (b) at least one senior must be chosen?
   (c) no more than one freshman may be chosen?

(d) two juniors and two seniors must be chosen?

(e) one member of each class must be chosen?

Solution:

(a) Since we're only interested in unordered groups of 4 people chosen from among the 18 club members, there are $\binom{18}{4} = 3060$ different choices possible.

(b) There are $\binom{6}{0} \cdot \binom{12}{4}$ delegations that contain no seniors, so there are $\binom{18}{4} - \binom{6}{0} \cdot \binom{12}{4} = 2565$ delegations that contain at least one senior.

(c) Either no freshmen are chosen, or exactly one freshman is chosen. Thus, the total number of delegations with at most one freshman is $\binom{4}{0} \cdot \binom{14}{4} + \binom{4}{1} \cdot \binom{14}{3} = 2457$.

(d) Choose two juniors, then two seniors: there are $\binom{3}{2} \cdot \binom{6}{2} = 45$ choices in all.

(e) Choose exactly one person from each class: there are $4 \cdot 5 \cdot 3 \cdot 6 = 360$ possibilities.

7. Six identically wrapped gift boxes are in a row. Two contain sackcloth and ash, three contain twenty dollar bills, and one contains a hundred dollar bill. How many possible arrangements of these gifts are there?

Solution:

Choose 2 of the 6 positions for the sackcloth and ashes, then 3 of the remaining 4 positions for the twenty dollar bills, then 1 of the remaining 1 positions for the hundred dollar bill. Thus we have $\binom{6}{2} \cdot \binom{4}{3} \cdot \binom{1}{1} = \binom{6}{2,3,1} = \dfrac{6!}{2! \cdot 3! \cdot 1!} = 60$ positions in all.

8. Five digit numbers are to be formed from the digits 1, 2, $\cdots$, 9. How many such numbers are there if

(a) there are no restrictions?

(b) no digit can be repeated?

(c) exactly one digit is repeated? (e.g. 31235)

Solution:

(a) $9^5 = 59,049$

(b) $P_{9,5} = 15,120$

(c) First, pick the digit that is to be repeated. Then pick two of the five slots to put the repeated digits in. Finally, from the remaining 8 digits choose 3 for the remaining spots. We have $9\binom{5}{2}P_{8,3} = 30240$ possibilities in all.

9. Twelve antennae are arranged in a row. The antennae are identical, except that three of them are not functioning. Suppose the array of antennae will still function correctly provided that none of the defective antennae are adjacent to each other. How many functioning arrangements of the twelve antennae are there?

Solution:

First, arrange the 9 working antennae in a row. Since the antennae are identical, there is only one way to do this. Now there are 10 possible positions for the remaining 3 defective antennae to go. Choose 3 of these (order does not matter). Thus, there are $\binom{10}{3} = 120$ possibilities in total.

10. Starting at point A in the graph below, you can move at each step either one unit to the right, or one unit up.

   (a) How many such paths from A to B are there?

   (b) How many such paths from A to B also pass through the point P?
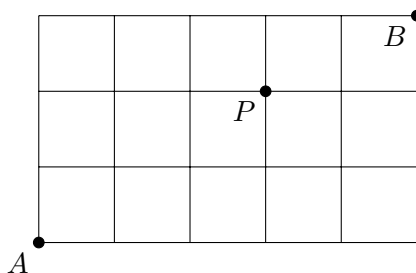
Solution:

   (a) Each path is specified by a string of R's and U's, indicating whether we move right or up at a given stage. (For example, $RRRRRUUU$ is one such string.) All such strings are of length 7, and must contain exactly 5 R's and 3 U's. So there are $\binom{8}{5} \cdot \binom{3}{3} = 56$ paths.

   (b) By the same logic as in part (a), there are $\binom{5}{3}$ paths from A to P, and for each of these, a further $\binom{3}{2}$ paths from P to B. So there are $\binom{5}{3} \cdot \binom{3}{2} = 30$ paths in all.