On my honor, I have neither given nor received any aid on this work, nor am I aware of any breach of the Honor Code that I shall not immediately report.

Pledged: ______________________________

Print Name: ______________________________

1. In each case, describe the sample space of the random experiment precisely. If the sample space is finite, find its cardinality. Also, if the sample space is finite, is it reasonable to regard the outcomes as equally likely?
   (a) Toss a pair of fair dice, and record the outcome.
   (b) Toss a pair of fair dice, and record the sum that appears.
   (c) Toss a fair die repeatedly until a 6 appears.

Solution:

(a) $S = \{(i, j) : i, j \in \{1, 2, 3, 4, 5, 6\}\}$, $|S| = 36$, S is an equally likely sample space.

(b) $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $|S| = 11$, S is not an equally likely sample space.

(c) $S = \{(x_1, x_2, \ldots, x_n) : n \in \mathbb{N},\ x_k \in \{1, 2, 3, 4, 5\}$ for $k = 1, 2, \cdots, n-1$, and $x_n = 6\}$, S is infinite.

2. An urn contains 4 red balls and 3 black balls. Draw a set of three balls from this urn "at random."
   (a) If all the balls are considered distinguishable (i.e. there's a first red ball, a second red ball, a first black ball, and so on), what is the cardinality of the sample space?
   (b) If balls of the same color are indistinguishable, how many outcomes are there in the sample space? Is it reasonable to consider these outcomes as equally likely?

Solution:

(a) $S = \{E \subseteq \{1, 2, 3, 4, 5, 6, 7\} : |E| = 3\}$, so $|S| = \binom{7}{3} = 35$.

(b) If balls of the same color are indistinguishable, and if the order in which the balls are drawn is considered irrelevant, then the result of the experiment is completely specified by the number of red (or black) balls drawn. Hence $S = \{0, 1, 2, 3\}$, and $|S| = 4$. The outcomes in S are not equally likely.

If the order in which the balls are drawn is to be taken into account, then we can take S to be the set of all strings of length 3 while contain only the letters R and B. (I.e. RRB would indicate that the first two balls drawn were red, and the third was black.) In this case, $|S| = 8$, and again, the outcomes are not equally likely.

3. Two dice are thrown. Let E be the event that the sum is even, let F be the event that at least one of the dice lands on 1, and let G be the event that the sum is 5. Describe each of the following events:
   (a) EF
   (b) $E \cup F$
   (c) FG
   (d) EF^c
   (e) EFG

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Solution:

(a) EF is the event that the sum is even, and at least one of the dice is 1. So

$$EF = \{(1,1),(1,3),(1,5),(3,1),(3,5)\}$$

(b) $E \cup F$ is the event that either the sum is even, or at least one of the dice is 1.

(c) $FG = \{(1,4),(4,1)\}$.

(d) EF^c is the event that the sum is even, and neither of the dice is 1.

(e) $EFG = \emptyset$

4. Toss a pair of six-sided dice repeatedly. Consider the following events:

$$\begin{aligned} A_n &= \{\text{the sum is 7 on toss number } n\} \\ B_n &= \{\text{both dice land on the same number on toss } n\} \\ C_n &= \{\text{the minimum of the two dice is 1 on toss } n\} \end{aligned}$$

Express the following events using complements, unions, and intersections of the above events:

(a) A sum of 7 occurs on or before the fourth toss.
(b) The first time the sum is 7 is on toss number four.
(c) A sum of 7 does not occur during the first four tosses.
(d) You roll "snake eyes" on toss number two. ("Snake eyes" means that both dice land on 1.)

Solution:

(a) $\displaystyle\bigcup_{n=1}^{4} A_n$

(b) $A_1^c A_2^c A_3^c A_4$

(c) $\displaystyle\left(\bigcup_{n=1}^{4} A_n\right)^c = A_1^c A_2^c A_3^c A_4^c$

(d) $B_2 C_2$

5. Suppose that A and B are events in a probability space, with $P(A) = 0.6$, $P(B) = 0.5$, and $P(AB) = 0.3$. Find the following:

(a) $P(AB^c)$
(b) $P(A^c B)$
(c) $P((A \cup B)^c)$
(d) $P(A^c \cup B^c)$

Solution:

(a) $P(AB^c) = 0.3$

(b) $P(A^c B) = 0.2$

(c) $P((A \cup B)^c) = 0.2$

(d) $P(A^c \cup B^c) = 0.7$

6. An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. These classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all three classes.
   (a) Choose a student at random. What is the probability that he or she is not in any of these language classes?
   (b) Choose a student at random. What is the probability that he or she is taking exactly one language class?
   (c) Choose two students at random. What is the probability that at least on of them is taking a language class?

Solution:
(a) $\dfrac{50}{100}$

(b) $\dfrac{32}{100}$

(c) $\dfrac{\binom{50}{1}\cdot\binom{50}{1}+\binom{50}{2}\cdot\binom{50}{0}}{\binom{100}{2}} = \dfrac{149}{198} \approx 0.7525$

7. A pair of fair dice are rolled.
   (a) Let X be the sum of the spots that appear. Find the probability mass function for X.
   (b) Let M be the maximum of the two dice. Find the probability mass function for M.
   (c) Find the probability that the dice land on different values.

Solution:
(a)

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(b)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

(c) $\dfrac{36-6}{36} = \dfrac{30}{36} \approx 0.8333$

8. A five-card hand is dealt from a well-shuffled deck of ordinary playing cards. What is the probability of being dealt
   (a) a flush? (A flush is a hand in which all five cards are of the same suit.)
   (b) one pair? (This occurs if the cards have denominations a, a, b, c, and d, where a, b, c, and d are all distinct.)
   (c) two pairs? (This occurs if the cards have denominations a, a, b, b, and c, where a, b, and c are all distinct.)
   (d) three of a kind? (This occurs if the cards have denominations a, a, a, b, and c, where a, b, and c are all distinct.)

Solution:
(a) $\dfrac{\binom{4}{1}\cdot\binom{13}{5}}{\binom{52}{5}} = \dfrac{33}{16660} \approx 0.001981$

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(b) $\dfrac{\binom{13}{1}\cdot\binom{4}{2}\cdot\binom{12}{3}\cdot\binom{4}{1}^3}{\binom{52}{5}} = \dfrac{352}{833} \approx 0.4226$

(c) $\dfrac{\binom{13}{2}\cdot\binom{4}{2}^2\cdot\binom{11}{1}\cdot\binom{4}{1}}{\binom{52}{5}} = \dfrac{198}{4165} \approx 0.04754$

(d) $\dfrac{\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{2}\cdot\binom{4}{1}^2}{\binom{52}{5}} = \dfrac{88}{4165} \approx 0.02113$

9. An urn contains 10 white balls and 5 black balls. Four balls are drawn randomly from the urn, without replacement. Let X be the number of black balls drawn. Find the probability mass function for X.

Solution:

$$P(X = x) = \frac{\binom{5}{x}\cdot\binom{10}{4-x}}{\binom{15}{4}}, \quad x = 0, 1, 2, 3, 4.$$

10. A pair of fair dice are tossed repeatedly until they show a sum of either 7 or 11. Let N be the number of tosses required. It can be shown that the probability mass function for N is of the form

$$P(N = n) = C\left(\frac{7}{9}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

(a) Find the constant C.

(b) How many tosses n are required for the probability to be at least $1/2$ that either a 7 or an 11 occurred on or before toss n? In other words, find the minimum value of n such that $P(N \leq n) \geq 1/2$.

Solution:

(a) We must have

$$1 = \sum_{n=1}^{\infty} P(N = n) = \sum_{n=1}^{\infty} C\left(\frac{7}{9}\right)^{n-1}.$$

Using the geometric series, we find

$$1 = \frac{C}{1 - 7/9},$$

or $C = 2/9$. So the probability mass function is

$$P(N = n) = \frac{2}{9}\left(\frac{7}{9}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

(b) $n = 3$ suffices, for we have

$$P(N \leq 3) = \sum_{n=1}^{3} P(N = n) = \sum_{n=1}^{3} \frac{2}{9}\left(\frac{7}{9}\right)^{n-1} = \frac{386}{729} \approx 0.5295$$