On my honor, I have neither given nor received any aid on this work, nor am I aware of any breach of the Honor Code that I shall not immediately report.

Pledged: ______________________________

Print Name: ______________________________

1. Suppose A and B are events in a probability space, with $P(A) = 0.3$ and $P(B) = 0.6$.
   (a) Given that A and B are independent, find the probability of $A \cup B$.
   (b) Given that $P(B|A) = 0.8$, find the probability of $A \cup B$.
   (c) Given that $P(A \cup B) = 0.8$, find the conditional probability of B given A.

Solution:

(a) $P(A \cup B) = P(A) + P(B) - P(AB)$. But $P(AB) = P(A)P(B)$ by independence, so $P(A \cup B) = 0.3 + 0.6 - 0.3 \cdot 0.6 = 0.72$

(b) If $P(B|A) = 0.8$, then $P(AB) = P(A)P(B|A) = 0.3 \cdot 0.8 = 0.24$. Hence $P(A \cup B) = 0.3 + 0.9 - 0.24 = 0.66$.

(c) In this case $P(AB) = 0.3 + 0.6 - 0.8 = 0.1$, so $P(B|A) = 0.1/0.3 = 1/3$

2. Roll a pair of fair dice. What is the expected value of the product of the two dice?

Solution:

Let X be the value on the first die, and Y be the value on the second. The joint mass function for X and Y is $P(X = x, Y = y) = 1/36$, for $1 \le x, y \le 6$. Hence

$$\begin{aligned} E[XY] &= \sum_{x=1}^{6}\sum_{y=1}^{6} xyP(X = x, Y = y) \\ &= \frac{1}{36}\left(\sum_{x=1}^{6} x\right)\left(\sum_{y=1}^{6} y\right) \\ &= \frac{21^2}{36} = \left(\frac{7}{2}\right)^2 \end{aligned}$$

3. Recall that a roulette wheel has 38 slots, 18 of which are red, 18 are black, and 2 are green. You have i dollars in your pocket, where i is an integer less than or equal to 100. You have resolved to bet one dollar on red repeatedly, until you either go bust, or you reach your goal of having \$100.
   (a) Let P_i be your probability of reaching your \$100 goal, given that you start with i dollars in your pocket, where $0 \le i \le 100$. Find a formula for P_i.
   (b) How large must your initial fortune be in order for you to have a better than even chance of reaching your \$100 goal?
   (c) Make an accurate plot of P_i as a function of i (you may use software).

 Question 3 continues on next page…

Solution:

(a) This is the Gambler's Ruin problem with $p = 18/38$ and $N = 100$. From our previous work,

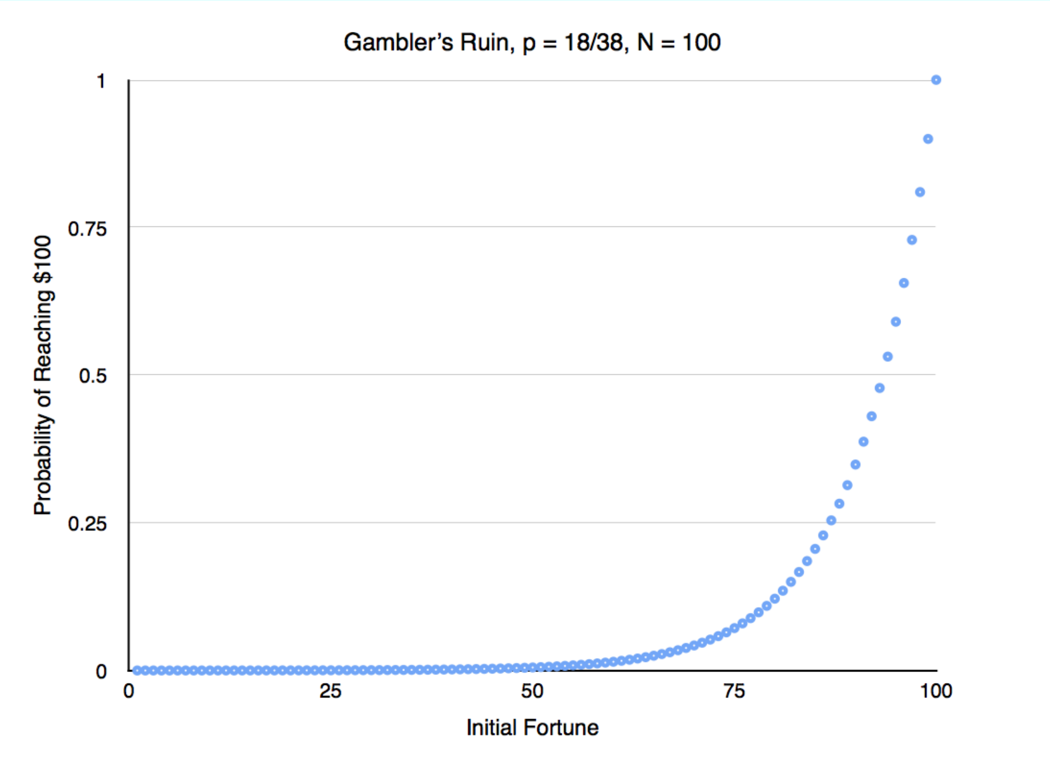
$$P_i = \frac{1 - (10/9)^i}{1 - (10/9)^{100}}$$

(b) Solving $P_i \geq 1/2$ gives

$$i \geq \frac{\ln(1/2 + (10/9)^{100}/2)}{\ln(10/9)} \approx 93.42$$

Hence, one would need to start with at least \$94 in order to stand a better-than-even chance of reaching \$100!

(c) Plot of P_i versus i, $0 \leq i \leq 100$:



4. Here is a game: there are three coins in an hat. Two are fair, while the third is two-headed. The player gets to choose a coin at random from the hat, and then flip it three times. Each time the flip results in heads, the player is paid \$1.

(a) Let X be the player's winnings in this game. Find the probability mass function for X.

(b) Suppose you want to charge people money to play this game. What is the minimum amount you'd need to charge in order to break even (or better) in the long run?

Solution:

(a) Let F be the event that the fair coin is selected. Then

$$\begin{aligned} P(X = x) &= P(F)\,P(X = x|F) + P(F^c)\,P(X = x|F^c) \\ &= \begin{cases} \frac{2}{3}\binom{3}{x}\left(\frac{1}{2}\right)^3 & \text{if } 0 \leq x \leq 2 \\ \frac{2}{3}\binom{3}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3}\cdot 1 & \text{if } x = 3 \end{cases} \end{aligned}$$

Tabulated, the probability mass function is

 Question 4 continues on next page...

x	0	1	2	3
$P(X=x)$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{5}{12}$

(b) $E[X] = 24/12 = 2$, so one would need to charge at least \$2.00 per customer to break even or better in the long run.

5. Let T have the geometric distribution with parameter p. Compute each of the following expected values:
   (a) $E\left[(3T-2)^2\right]$
   (b) $E\left[2^T\right]$
   (c) $E\left[e^{tT}\right]$, where t is an arbitrary real number.

Solution:

(a) We have seen that if T has the Geometric(p) distribution, then $E[T] = 1/p$, and $E\left[T^2\right] = (2-p)/p^2$. Thus

$$\begin{aligned} E\left[(3T-2)^2\right] &= E\left[9T^2 - 12T + 4\right] \\ &= 9E\left[T^2\right] - 12E[T] + 4 \\ &= 9\frac{2-p}{p^2} - 12\frac{1}{p} + 4 \\ &= \frac{18}{p^2} - \frac{21}{p} + 4 \end{aligned}$$

(b)

$$\begin{aligned} E\left[2^T\right] &= \sum_{k=1}^{\infty} 2^k (1-p)^{k-1} p \\ &= 2p \sum_{k=1}^{\infty} (2(1-p))^{k-1} \\ &= \frac{2p}{1-2(1-p)} \\ &= \frac{2p}{2p-1} \end{aligned}$$

Note however that the series only converges if $2(1-p) < 1$, or equivalently, $p > 1/2$. Otherwise, the expected value is infinite!

(c)

$$\begin{aligned} E\left[e^{tT}\right] &= \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} p \\ &= pe^t \sum_{k=1}^{\infty} \left(e^t(1-p)\right)^{k-1} \\ &= \frac{pe^t}{1-(1-p)e^t} \end{aligned}$$

Note however, that, as a function of t, the series only converges for $e^t(1-p) < 1$, or equivalently, $t < -\ln(1-p)$.

6.

(a) Toss a pair of fair dice repeatedly. Find the probability that a sum of 11 appears before a sum of 7 appears.

(b) Let E and F be mutually exclusive events of some random experiment. Suppose that independent trials of this experiment are run. Find the probability that the event E occurs before the event F. (Your answer will of course depend on $P(E)$ and $P(F)$.)

Solution:

(a) Please refer to part (b) below.

(b) Let $G = (E \cup F)^c$, and let E_i, F_i and G_i be the events that E, F, or G occurs on trial i, respectively. Then

$$\begin{aligned}\{E \text{ occurs before } F\} &= E_1 \cup G_1E_2 \cup G_1G_2E_3 \cup \cdots \\ &= \bigcup_{n=1}^{\infty} \left(\bigcap_{i=1}^{n-1} G_i \right) \cap E_n\end{aligned}$$

The events in the union are clearly mutually exclusive. Moreover, by independence

$$P\left(\left(\bigcap_{i=1}^{n-1} G_i \right) \cap E_n \right) = (P(G))^{n-1} P(E)$$

Hence

$$\begin{aligned}P(E \text{ before } F) &= \sum_{n=1}^{\infty} (P(G))^{n-1} P(E) \\ &= P(E) \frac{1}{1 - P(G)} \\ &= \frac{P(E)}{P(E) + P(F)}\end{aligned}$$

Returning to part (a) now, the probability that an 11 appears before a 7 when a pair of fair dice are repeatedly tossed is

$$\frac{2/36}{2/36 + 6/36} = \frac{1}{4}$$

Alternate Solution:
Let $G = (E \cup F)^c$. Notice that $\{E, F, G\}$ is a mutually exclusive and exhaustive set of events on any single trial. Now, let A be the event that E occurs before F does in the sequence of trials. Condition on the outcome of the first trial – either E, F, or G must occur:

$$P(A) = P(E) P(A|E) + P(F) P(A|F) + P(G) P(A|G)$$

Now clearly $P(A|E) = 1$, $P(A|F) = 0$. Also, if neither E nor F occurs on trial 1, then by independence the process "restarts", i.e. $P(A|G) = P(A)$. Hence we have

$$\begin{aligned}P(A) &= P(E) \cdot 1 + P(F) \cdot 0 + P(G) P(A) \\ &= P(E) + P(G) P(A)\end{aligned}$$

Solving this for $P(A)$ gives

$$P(A) = \frac{P(E)}{1 - P(G)} = \frac{P(E)}{P(E) + P(F)}$$

7. Consider a Bernoulli trials process with success probability p. Let

$$C = \text{the first time two consecutive successes occur.}$$

For example, if the sequence of successes and failures is $SFFSFSSF\cdots$, then $C = 7$. Clearly the possible values of C are $S_C = \{2, 3, 4, \cdots\}$. Find $P(C = 7)$.

Solution:

Let $P_n = P(C = n)$. Clearly $P_1 = 0$, $P_2 = p^2$, and $P_3 = (1-p)p^2$. Moreover, for $n \geq 4$ we have

$$P_n = P(\text{no consecutive successes occur in trials 1 through } n-3) \cdot (1-p)p^2$$

Let Q_k be the probability that no consecutive successes occur in trials 1 through k. Then we have $P_n = Q_{n-3}(1-p)p^2$. Now clearly $Q_1 = 1$, $Q_2 = 1 - p^2$, and conditioning on the outcome of the last trial, we see that for $k \geq 3$ we have

$$Q_k = (1-p)Q_{k-1} + p(1-p)Q_{k-2}$$

Iterating the recursion, we get

$$\begin{aligned}
Q_3 &= (1-p)(1-p^2) + p(1-p) \\
Q_4 &= (1-p)\left[(1-p)(1-p^2) + p(1-p)\right] + p(1-p)[1-p^2] \\
&= 2p^3 - 3p^2 + 1 \\
&= (p-1)^2(2p+1)
\end{aligned}$$

Hence

$$\begin{aligned}
P_7 &= Q_4(1-p)p^2 \\
&= (1-p)^3 p^2 (2p+1)
\end{aligned}$$