ON MY HONOR, I HAVE NEITHER GIVEN NOR RECEIVED ANY AID ON THIS WORK, NOR AM I AWARE OF ANY BREACH OF THE HONOR CODE THAT I SHALL NOT IMMEDIATELY REPORT.

Pledged: ________________________________

Print Name: ________________________________

1. Let X have the *discrete uniform distribution* on $\{1, 2, \cdots, n\}$, i.e. $P(X = k) = 1/n$ for $k = 1, 2, \cdots n$. Find the mean and variance of X.

Solution:
For the mean, we have

$$E[X] = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$$

For the variance, we have $\mathrm{Var}(X) = E\left[X^2\right] - E[X]^2$, and

$$E\left[X^2\right] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n} = \frac{1}{n}\sum_{k=1}^{n} k^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

So

$$\mathrm{Var}(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$

2. Let X be a discrete random variable with probability mass function $p(x)$. The *entropy* of X is the quantity $H(X)$ defined by

$$H(X) = -E\left[\log_2(p(X))\right]$$

Like variance, entropy can be considered as a measure of uncertainty in a distribution. (In fact the units of entropy are *bits*.) Find the entropy of the Bernoulli(p) random variable. Plot $H(X)$ as a function of p, and comment.

Solution:
X is Bernoulli(p), so its mass function $p(x)$ is given by

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

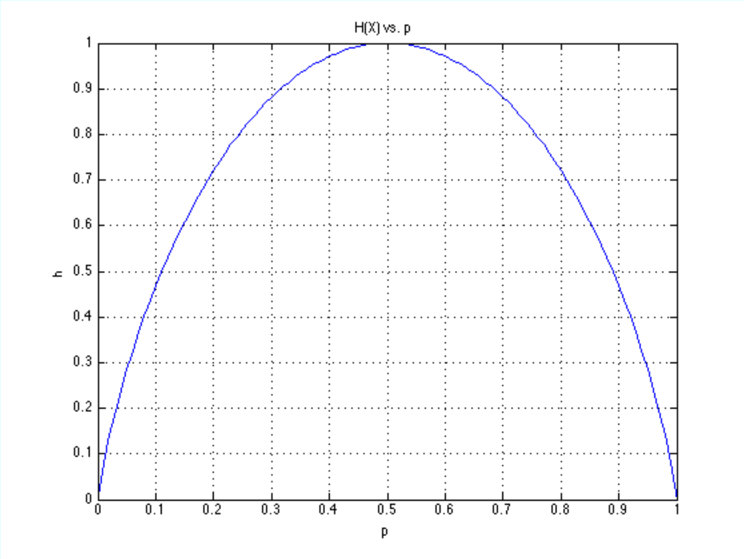
Hence,

$$\begin{aligned} E\left[\log_2(p(X))\right] &= \sum_{x=0}^{1} \log_2(p(x)) \cdot p(x) \\ &= \log_2(1-p)\cdot(1-p) + \log_2(p)\cdot p \end{aligned}$$

Thus

$$H(X) = -E\left[\log_2(p(X))\right] = -(1-p)\log_2(1-p) - p\log_2(p)$$

The plot of $H(X)$ as a function of p is given below. Notice that the entropy (or uncertainty) is maximized when $p = 1/2$ (when the entropy is 1 bit), and minimized when $p = 0$ or $p = 1$ (when the entropy is zero bits). This makes sense intuitively: a fair coin is more unpredictable than any biased coin, while a two-headed coin ($p = 1$) or a two-tailed ($p = 0$) coin is completely predictable.



3. Let X have possible values $\{1,2\}$, let Y have possible values $\{1,2,3\}$, and suppose the joint mass function for (X,Y) is $p(x,y)=C(x+y)$ where $x\in\{1,2\}$, $y\in\{1,2,3\}$, and C is a constant.
   (a) Find the value of C.
   (b) Find the marginal mass functions for X and Y.
   (c) Are X and Y independent? Explain.

Solution:

(a) We must have $\sum_{x=1}^{2}\sum_{y=1}^{3}p(x,y)=1$. Thus

$$1=\sum_{x=1}^{2}\sum_{y=1}^{3}C(x+y)=C\left(\sum_{x=1}^{2}\sum_{y=1}^{3}x+\sum_{y=1}^{3}\sum_{x=1}^{2}y\right)$$
$$=C\left(3\sum_{x=1}^{2}x+2\sum_{y=1}^{3}y\right)=C\left(3\cdot 3+2\cdot 6\right)=21C$$

So $C=1/21$.

(b)

$$p_X(x)=\sum_{y=1}^{3}p(x,y)=\frac{1}{21}\sum_{y=1}^{3}(x+y)=\frac{3x+6}{21},\quad x=1,2$$
$$p_Y(y)=\sum_{x=1}^{2}p(x,y)=\frac{1}{21}\sum_{x=1}^{2}(x+y)=\frac{3+2y}{21},\quad y=1,2,3$$

(c) Note that

$$p(1,1)=\frac{1+1}{21}=\frac{2}{21}$$
$$p_X(1)\cdot p_Y(1)=\frac{9}{21}\cdot\frac{5}{21}=\frac{5}{49}$$

Hence, $p(x,y)\neq p_X(x)\cdot p_Y(y)$, and thus X and Y are not independent.

4. Deal a five-card hand from well-shuffled deck, and let X be the number of hearts in the hand, and Y be the number of clubs. Find the joint probability mass function for X and Y.

Solution:

$$p(x,y) = P(X = x, Y = y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{5-x-y}}{\binom{52}{5}}, \quad x, y \geq 0,\ x + y \leq 5$$

5. Consider a Bernoulli trials process with success probability p. Let X_n be the number of successes up to time n. Find the joint probability mass function for X_m and X_{m+n}. (Hint: Find $P(X_m = i, X_{m+n} = i + j)$ using the multiplication principle for conditional probabilities.)

Solution:

$$P(X_m = i, X_{m+n} = i + j) = P(X_m = i)\, P(X_{m+n} = i + j | X_m = i)$$

Clearly $P(X_m = i) = \binom{m}{i} p^i (1-p)^{m-i}$. Also, given that $X_m = i$, we have $X_{m+n} = i+j$ if and only if there are exactly j successes in trials $m + 1$ through $m + n$. By independence this happens with probability $\binom{n}{j} p^j (1-p)^{n-j}$. Hence

$$\begin{aligned} P(X_m = i, X_{m+n} = i + j) = \binom{m}{i} p^i (1-p)^{m-i} \binom{n}{j} p^j (1-p)^{n-j} \\ \binom{m}{i}\binom{n}{j} p^{i+j} (1-p)^{m+n-(i+j)}, \end{aligned}$$

for $0 \leq i \leq m$, $0 \leq j \leq n$.

6. Suppose that X and Y are independent discrete random variables. Show that $E[XY] = E[X]\, E[Y]$.

Solution:

$$\begin{aligned} E[XY] &= \sum_{x \in S_X} \sum_{y \in S_Y} xyP(X = x, Y = y) \\ &= \sum_{x \in S_X} \sum_{y \in S_Y} xyP(X = x)\, P(Y = y) \quad \text{(by independence)} \\ &= \sum_{x \in S_X} xP(X = x) \left(\sum_{y \in S_Y} yP(Y = y) \right) \\ &= \left(\sum_{y \in S_Y} yP(Y = y) \right) \left(\sum_{x \in S_X} xP(X = x) \right) \\ &= E[Y]\, E[X] \end{aligned}$$

7. The probability generating function for an integer-valued random variable X is

$$P(z) = \frac{1}{6} + \frac{1}{3}z + \frac{1}{2}z^2$$

(a) Find the probability mass function for X.

(b) Find the moment-generating function for X.

 Question 7 continues on next page. . .

(c) Suppose Y and Z are random variables with the same probability generating function as X, and that $\{X, Y, Z\}$ is independent. Find the probability mass function for $S = X + Y + Z$.

Solution:

(a)

x	0	1	2
$P(X = x)$	1/6	1/3	1/2

(b)

$$M_X(t) = \frac{1}{6} + \frac{1}{3}e^t + \frac{1}{2}e^{2t}$$

(c) The probability generating function for S is

$$\begin{aligned} P_S(z) &= \left(\frac{1}{6} + \frac{1}{3}z + \frac{1}{2}z^2\right)^3 \\ &= \frac{1}{216} + \frac{1}{36}z + \frac{7}{72}z^2 + \frac{11}{54}z^3 + \frac{7}{24}z^4 + \frac{1}{4}z^5 + \frac{1}{8}z^6 \end{aligned}$$

Hence the mass function for S is

s	0	1	2	3	4	5	6
$P(S = s)$	1/216	1/36	7/72	11/54	7/24	1/4	1/8

8. The probability generating function for a certain integer-valued random variable X is $P(z) = e^{\lambda(z-1)}$, where λ is a positive constant.

(a) Find the probability mass function for X.

(b) Find the mean and variance of X.

Solution:

(a)

$$P(z) = e^{-\lambda}e^{\lambda t} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}z^k$$

Hence the probability mass function for X is

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N}$$

(This is called the *Poisson* distribution with parameter λ.)

(b) The moment generating function for X is $M(t) = e^{\lambda\left(e^t - 1\right)}$. So $M'(t) = M(t)\lambda e^t$, and thus

$$E[X] = M'(0) = \lambda.$$

Also

$$M''(t) = M'(t)\lambda e^t + M(t)\lambda e^t,$$

so $E\left[X^2\right] = M''(0) = \lambda^2 + \lambda$. Therefore

$$\mathrm{Var}(X) = E\left[X^2\right] - E[X]^2 = \lambda$$