Section 1.2, problem 2. Let

\[ B = \{ \text{customers who insure 2 or more cars} \} \]
\[ C = \{ \text{customers who insure a sports car} \} \]
\[ D = \{ \text{customers who insure 2 or more cars, including a sports car} \} = BC \]

Then

\[ \{ \text{customers who insure exactly 1 car, and not a sports car} \} = (B \cup C)' \]

Given that \( P(B) = 0.85 \), \( P(C) = 0.23 \), and \( P(BC) = 0.17 \), it follows that

\[ P(B \cup C) = P(B) + P(C) - P(BC) = 0.91, \]

so \( P((B \cup C)') = 1 - 0.91 = 0.09. \)

Section 1.2, problem 4.

(a) The sample space in set-builder notation is \( S = \{(x_1,x_2,x_3,x_4) : x_i \in \{H,T\} \text{ for } i = 1,2,3,4\} \). Note that \(|S| = 2^4\). Listing the elements of \( S \) gives

\[ S = \{TTTT, TTTH, TTHT, TTHH, THTT, THTH, THHT, THHH, HTTT, HTTH, HTHT, HTHH, HHTT, HHTH, HHHT, HHHH\} \]

(b) (i) \( P(\text{at least 3 heads}) = \frac{1}{2^4} \left( \binom{4}{3} \cdot \binom{1}{1} + \binom{4}{0} \cdot \binom{0}{0} \right) = \frac{5}{16} \)

(ii) \( P(\text{at least 3 heads AND at most 2 heads}) = \frac{1}{2^4} \left( \binom{4}{0} \cdot \binom{4}{0} + \binom{4}{3} \cdot \binom{1}{1} + \binom{4}{2} \cdot \binom{2}{2} \right) = \frac{11}{16} \)

(iv) \( P(\text{at least 3 heads AND heads on toss 3}) = \frac{1}{2^4} \left( \binom{2}{2} \cdot \binom{1}{1} + 1 \cdot \binom{3}{0} \cdot \binom{0}{0} \right) = \frac{4}{16} \)

(v) \( P(\text{1 head and 3 tails}) = \frac{1}{2^4} \left( \binom{4}{1} \cdot \binom{3}{3} \right) = \frac{4}{16} \)

(vi) \( P(\text{at least 3 heads OR a head on toss 3}) = 1 - P(2 \text{ or fewer heads AND a tail on toss 3}) = 1 - \frac{1}{2^4} \left( \binom{0}{0} + \binom{1}{1} + \binom{2}{2} \right) = 1 - \frac{7}{16} = \frac{9}{16} = 9/16. \)

(vii) Note that \( D \subseteq B \), so \( B \cap D = D \), and \( P(D) = \frac{1}{2^4} \binom{4}{1} = 4/16 \)

Section 1.2, problem 6. We have \( A = \{ \text{die shows 3 on roll 1} \} \), and \( B = \{ \text{first 3 occurs on roll 2 or later} \} \).

(a) \( P(A) = 1/6. \)

(b) Since \( B = A' \), it follows that \( P(B) = 1 - 1/6 = 5/6. \)

(c) Since \( B = A' \), we have \( A \cup B = S \), so \( P(A \cup B) = P(S) = 1. \)

Section 1.2, problem 8. We have \( P(A) = 0.4, P(B) = 0.5 \), and \( P(AB) = 0.3 \). The Venn diagram below gives the situation:
Section 1.2, problem 10. Let \( L = \{ \text{patient requires lab work} \} \) and \( R = \{ \text{patient requires a referral} \} \). We have \( P (L) = 0.41 \), \( P (R) = 0.53 \), and \( P (L' R') = 0.21 \). It follows that \( P (L \cup R) = P (L) + P (R) - P (LR) \), or 0.79 = 0.41 + 0.53 - \( P (LR) \). Solving yields \( P (LR) = 0.15 \).

Section 1.2, problem 16. (a) \( S = \{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \} \).

(b) (i) \( P (\text{sum} = 3) = P (\{1, 2\}) = 1/10 \).

(ii) \( P (3 \leq \text{sum} \leq 8) = 1 - P (\{4, 5\}) = 9/10 \).

Section 1.2, problem 18. Let \( X \) be the point chosen randomly and uniformly from \( S = [-r, r] \), and let \( E \) be the event that the corresponding perpendicular has length no more than \( r/2 \). The event \( E \) is shown in red in the figure below.

So \( E = \{ x \in S : x \geq \sqrt{3}r/2 \text{ or } x \leq -\sqrt{3}r/2 \} \). Since \( X \) is chosen uniformly from \( S = [-r, r] \), we have

\[
P (E) = \frac{|E|}{|S|} = \frac{2 (r - \sqrt{3}r/2)}{2r} = 1 - \sqrt{3}/2 \approx 0.1340.
\]

Section 1.3, problem 6. (a) \( \binom{4}{4} \cdot \binom{6}{3} = 80 \).

(b) \( \binom{4}{4} \cdot 2^6 = 256 \).

(c) This is equal to the number of non-negative integer vectors of length \( \ell = 4 \) whose entries sum to \( n = 3 \), which is

\[
\binom{3 + 4 - 1}{4 - 1} = \binom{6}{3} = 20.
\]
Section 1.3, problem 10. First note that the tournament lasts for at most 5 games. Represent a particular outcome as a string of length 5, in which position $i$ is an “H” if Hope wins game $i$, a “C” if Calvin wins game $i$, and an “x” if game $i$ is not played. So for example, $HCHHx$ represents the outcome in which Hope wins game 1, Calvin wins game 2, and then Hope wins games 3 and 4, and thus the tournament. (Game 5 would not be played in this case.)

How many such sequences are there in which Hope wins the tournament? Such a sequence is specified as follows:

1. Choose 3 of the 5 positions for the $H$’s.
2. Fill in any blank slots before the last $H$ with $C$’s.
3. Fill in any blank slots after the last $H$ with $x$’s.

By the multiplication principle there are $\binom{5}{3} \cdot 1 \cdot 1 = 10$ such sequences. By symmetry, there are also 10 sequences in which Calvin wins. So there are 20 different possible outcomes for the tournament.

Section 1.3, problem 12. $\binom{3}{1} \cdot 2^{12} = 12,288$.

Section 1.3, problem 14. Here is a combinatorial proof of Pascal’s identity. The idea of the proof is simple: the left-hand side of the equation counts the number of subsets of size $r$ in a set of size $n$. We will show that the right-hand side counts the same thing.

Consider a set with exactly $n$ elements. Pick one of these elements, and call it “Fred”. Now any subset of size $r$ either contains “Fred” or it doesn’t. The number of subsets of size $r$ that contain “Fred” is

$$1 \cdot \binom{n-1}{r-1}$$

The number of subsets of size $r$ that do not contain “Fred” is

$$\binom{n-1}{r}$$

Hence the total number of subsets of size $r$ is

$$\binom{n-1}{r-1} + \binom{n-1}{r}$$

But we already know that the total number of subsets of size $r$ is $\binom{n}{r}$. Therefore it must be the case that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Section 1.3, problem 17. The sample space $S$ of all possible 5-card poker hands has cardinality $|S| = \binom{52}{5}$.

(a) $P$ (four of a kind) = $\frac{\binom{13}{1} \binom{4}{1} \binom{12}{1} \binom{4}{1}}{\binom{52}{5}} \approx 0.0002401$

(b) $P$ (two of a kind AND three of a kind) = $\frac{\binom{13}{1} \binom{4}{2} \binom{12}{1} \binom{4}{3}}{\binom{52}{5}} \approx 0.001441$

(c) $P$ (three of a kind) = $\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1}^2}{\binom{52}{5}} \approx 0.02113$

(d) $P$ (two pair) = $\frac{\binom{13}{1} \binom{4}{2} \binom{11}{1} \binom{4}{1}}{\binom{52}{5}} \approx 0.04754$

(e) $P$ (one pair) = $\frac{\binom{13}{1} \binom{4}{1} \binom{12}{2} \binom{4}{3}^3}{\binom{52}{5}} \approx 0.4226$
Section 1.3, problem 20.

(a) \[ P(3 \text{ white hearts}) = \frac{\binom{19}{3} \binom{52-19}{3}}{\binom{52}{9}} \approx 0.2917 \]

(b) \[ P(3 \text{ whites, 2 tans, 1 pink, 1 yellow, 2 green}) = \frac{\binom{19}{3} \binom{10}{2} \binom{7}{1} \binom{6}{1} \binom{6}{2}}{\binom{52}{9}} \approx 0.006222 \]

Section 1.3, problem 22. This is the number of non-negative integer vectors of length \( \ell = 10 \) that sum to \( n = 36 \), which is

\[ \binom{10 + 36 - 1}{10 - 1} = \binom{45}{9} = 886,163,135 \]