4.1, 4.10

\[ f(x,y) = \begin{cases} \frac{3}{2} & \text{if } 0 \leq x \leq 1, x^2 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ F_x(x_0) = \mathbb{P}(X \leq x_0) \]

\[ = \int_{0}^{x_0} \left( \int_{y=x^2}^{1} \frac{3}{2} \, dy \right) \, dx \]

\[ = \int_{0}^{x_0} \frac{3}{2} (1-x^2) \, dx \]

\[ = \left[ \frac{3}{2} \left( x - \frac{1}{3} x^3 \right) \right]_{0}^{x_0} = \frac{3}{2} \left( x_0 - \frac{1}{3} x_0^3 \right) \]
\[ F_Y(y_0) = P(Y \leq y_0) \]
\[ = \int_{y=0}^{y_0} \left( \int_{x=0}^{\sqrt{3y}} \frac{3}{2} \, dx \right) \, dy \]
\[ = \int_{y=0}^{y_0} \frac{3}{2} \sqrt{y} \, dy = \frac{3}{2} \cdot \frac{2}{3} y^{3/2} \Big|_0^{y_0} = y_0^{3/2} \]

a) \[ P(0 \leq X \leq \frac{1}{2}) = F_X(\frac{1}{2}) \]
\[ = \frac{3}{2} \left( \frac{1}{2} - \frac{1}{3} \left( \frac{1}{2} \right)^3 \right) = \frac{11}{16} \]

b) \[ P\left( \frac{1}{2} \leq Y \leq 1 \right) = P\left( \frac{1}{2} \leq Y \right) \quad \text{since} \ Y \leq 1 \]
\[ = 1 - P(Y < \frac{1}{2}) \]
\[ = 1 - F_Y(\frac{1}{2}) \]
\[ = 1 - \left( \frac{1}{2} \right)^{3/2} = 1 - \frac{\sqrt{2}}{4} \]
c) \[ P \left( 0 \leq X \leq \frac{1}{2}, \frac{1}{2} \leq Y \leq 1 \right) = \int_{Y=1/2}^{Y=1} \left( \int_{X=0}^{X=1/2} \frac{3}{2} \, dx \right) \, dy \]
\[= \int_{Y=1/2}^{Y=1} \frac{3}{2} \cdot \frac{1}{2} \, dy = \frac{3}{4} \left( 1 - \frac{1}{2} \right) \]
\[= \frac{3}{8} \]

d) \[ P \left( X \leq \frac{1}{2}, Y \geq \frac{1}{2} \right) = \int_{Y=1/2}^{Y=1} \left( \int_{X=0}^{X=1/2} \frac{3}{2} \, dx \right) \, dy \]
\[= \int_{Y=1/2}^{Y=1} \frac{3}{2} \left( \frac{1}{2} - \frac{1}{2} \right) \, dy \]
\[= \frac{3}{2} \left( \frac{2}{3} \cdot \frac{3}{2} - \frac{1}{2} \cdot \frac{1}{2} \right) \bigg|_{Y=1/2}^{Y=1} \]
\[= \frac{3}{2} \left( \frac{2}{3} - \frac{1}{4} \right) \bigg|_{Y=1/2}^{Y=1} \]
\[= \frac{5 - 2\sqrt{2}}{8} \]
e) \ X \text{ and } Y \text{ are not independent, since } S_{X,Y} \text{ is not rectangular.}
4.1, #12

\[ f(x, y) = \begin{cases} 
    x + y & \text{if } 0 \leq x, y \leq 1 \\
    0 & \text{otherwise}
\end{cases} \]

Marginals

Marginal CDF for \( X \) is

\[ F_X(x_0) = P(X \leq x_0) = \int_{y=0}^{\infty} \left( \int_{x=0}^{x_0} x + y \, dx \right) \, dy \]

\[ = \int_{x=0}^{x_0} (xy + \frac{1}{2} y^2) \, dy \]

\[ = \left[ \frac{1}{2} x^2 y + \frac{1}{2} xy^2 \right]_0^{x_0} = \frac{1}{2} (x_0^2 + x_0) \]

So the marginal PDF of \( X \) is

\[ f_X(x) = F_X'(x) = \frac{1}{2} (2x + 1) = x + \frac{1}{2}, \quad 0 \leq x \leq 1 \]

Similarly, the marginal PDF of \( Y \) is

\[ f_Y(y) = y + \frac{1}{2}, \quad 0 \leq y \leq 1 \]
Note

\[ P_X(x) \cdot P_Y(y) = (x + \frac{1}{2})(y + \frac{1}{2}) \neq x + y = f(x, y) \]

So \( X \) and \( Y \) are not independent. They are identically distributed, however, because they have the same PDF's. In particular, they have the same means and variances.

\[ m_x = m_y = \int_0^1 x (x + \frac{1}{2}) \, dx = \left( \frac{1}{12} x^3 + \frac{1}{4} x^2 \right) \bigg|_0^1 = \frac{7}{12} \]

\[ E(X^2) = E(Y^2) = \int_0^1 x^2 (x + \frac{1}{2}) \, dx = \left( \frac{1}{4} x^4 + \frac{1}{6} x^3 \right) \bigg|_0^1 = \frac{5}{12} \]

\[ \sigma^2_x = \sigma^2_y = \frac{5}{12} - \left( \frac{7}{12} \right)^2 = \frac{11}{144} \]
4.1, #14

Let $E = \{ T_1 + T_2 > 10 \}$. Since the joint distribution is uniform on $S_{T_1, T_2}$, we have

$$P(E) = \frac{\text{Area}(E)}{\text{Area}(S_{T_1, T_2})}$$

Now

$$\text{Area}(E) = \frac{1}{2}(2)(2) = 2$$

and

$$\text{Area}(S_{T_1, T_2}) = 1.4 + \frac{1}{2}(8)4 = 20$$

So

$$P(E) = \frac{2}{20} = \frac{1}{10}$$

(See figure on next page.)
$4.1 \# 14 (cont\')$

The diagram shows a system $S_{T_1, T_2}$ with a shaded region. The equation $t_1 + 2t_2 = 14$ is labeled on the diagram, passing through the point $(6, 4)$. Another line $t_1 + t_2 = 10$ is also shown. The shaded region $E$ is defined by $t_1 + T_2 > 103$. 

Mathematically, the expression can be written as:

$$E = \exists T_1, T_2 : T_1 + T_2 > 103$$
5.3, #2

a) \( P(X_1 = 2, X_2 = 4) = P(X_1 = 2) \cdot P(X_2 = 4) \) (by independence)

\[
= \left( \frac{3}{4} \right) \left( \frac{1}{2} \right)^3 \left( \frac{5}{4} \right) \left( \frac{1}{2} \right)^5 \\
= 15 \cdot \left( \frac{1}{2} \right)^8 = \frac{15}{256}
\]

b) By independence, the MGF of \( X_1 + X_2 \) is

\[
M(t) = M_{X_1}(t) \cdot M_{X_2}(t)
\]

\[
= \left( \frac{1}{2} + \frac{1}{2} et \right)^3 \left( \frac{1}{2} + \frac{1}{2} et \right)^5 \\
= \left( \frac{1}{2} + \frac{1}{2} et \right)^8
\]

So, \( X_1 + X_2 \) has the Binomial \( (n=8, p=\frac{1}{2}) \) distribution.

So

\[
P(X_1 + X_2 = 7) = \left( \frac{8}{7} \right) \left( \frac{1}{2} \right)^8 = \frac{8}{256} = \frac{1}{32}
\]
$X_1$ and $X_2$ are IID, with mass function

$$P(x) = \begin{cases} \frac{x}{6} & \text{if } x \in \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

So,

$$P(X_1 + X_2 = 5) = \sum_{x=1}^{3} P(X_1 = x, X_2 = 5-x)$$

$$= \sum_{x=1}^{3} P(X_1 = x) P(X_2 = 5-x) \quad \text{(by independence)}$$

Note that the terms in the sum are non-zero if and only if

$$1 \leq x \leq 3 \quad \text{and} \quad 1 \leq 5-x \leq 3$$

or equivalently

$$1 \leq x \leq 3 \quad \text{and} \quad 5-3 \leq x \leq 5-1$$

or

$$\max(1, 5-3) \leq x \leq \min(3, 5-1)$$
\[ P(\sum_{i=1}^{2} X_i = S) = \sum_{\substack{x \in \min(S, S-1) \cap \mathbb{N} \cap \max(1, S-2) \cap \mathbb{N}}} \frac{x}{6} \cdot \frac{S-x}{6} \]

for \( S = \{2, 3, 4, 5, 6\} \). Tabulating this gives:

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(\sum_{i=1}^{2} X_i = S) )</td>
<td>( \frac{1}{36} )</td>
<td>( \frac{4}{36} )</td>
<td>( \frac{10}{36} )</td>
<td>( \frac{12}{36} )</td>
<td>( \frac{9}{36} )</td>
</tr>
</tbody>
</table>
5.4, #2

\[ X_1 \sim \text{Bin}(n_1, p), \quad X_2 \sim \text{Bin}(n_2, p) \]

\[ X_1 \perp X_2. \]

So the MGF for \( X_1 + X_2 \) is

\[
M(t) = E\left( e^{t(X_1+X_2)} \right)
\]

\[
= E\left( e^{tX_1} e^{tX_2} \right)
\]

\[
= E(e^{tX_1}) E(e^{tX_2}) \quad \text{(by independence)}
\]

\[
= (q + pet)^{n_1} (q + pet)^{n_2}
\]

\[
= (q + pet)^{n_1+n_2}
\]

This is the MGF for the \( \text{Bin}(n_1+n_2, p) \) distribution.

So \( X_1 + X_2 \sim \text{Bin}(n_1+n_2, p) \)
If $X$ is Poisson ($\lambda = \mu$), then the MGF of $X$ is

$$M_X(t) = e^{\mu(e^t-1)}$$

So, if $\{X_i : 1 \leq i \leq n\}$ is an independent family of RV's, with $X_i \sim \text{Poisson}(\lambda = \mu_i)$, $1 \leq i \leq n$, then the MGF for $Y = \sum_{i=1}^{n} X_i$ is

$$M(t) = E\left( \exp\left( t\sum_{i=1}^{n} X_i \right) \right)$$

$$= E\left( \prod_{i=1}^{n} e^{tX_i} \right)$$

$$= \prod_{i=1}^{n} E\left( e^{tX_i} \right) \quad \text{(by independence)}$$

$$= \prod_{i=1}^{n} e^{\mu_i(e^t-1)}$$

$$= \exp\left( \sum_{i=1}^{n} \mu_i(e^t-1) \right)$$

$$= \exp\left( \left( \sum_{i=1}^{n} \mu_i \right)(e^t-1) \right)$$
This is the MLE for the Poisson distribution with mean \( \sum_{\lambda=1}^{\infty} n_{\lambda} \). So \( Y = \sum_{\lambda=1}^{n} X_{\lambda} \) is Poisson \( (\lambda = \sum_{\lambda=1}^{\infty} n_{\lambda}) \).
5.4, #6

a) The MGF for $X_i$, $1 \leq i \leq 5$, is

$$M(t) = \frac{pet}{1 - qet}$$

Since $p = \frac{1}{5}$, $q = \frac{2}{5}$, we get

$$M(t) = \frac{\frac{1}{5}et}{1 - \frac{2}{5}et}$$

By independence, the MGF for $Y = \sum_{i=1}^{5} X_i$ is

$$M_Y(t) = \left( \frac{\frac{1}{5}et}{1 - \frac{2}{5}et} \right)^5 = \frac{\left( \frac{1}{5}et \right)^5}{(1 - \frac{2}{5}et)^5}$$

b) This is the MGF of the negative binomial distribution, with

$Y = 5$ and $p = \frac{1}{5}$. 
5.4, #10

a) \( M_X(t) = \sum_{x=0}^{3} e^{tx} P(X=x) = \frac{1}{4} (e^{t0} + e^{t1} + e^{t2} + e^{t3}) \)
\[ = \frac{1}{4} (1 + e^t + e^{2t} + e^{3t}) \]

Similarly

b) \( M_Y(t) = \frac{1}{4} (1 + e^{4t} + e^{8t} + e^{12t}) \)

So if \( W = X + Y \) and \( X \) and \( Y \) are independent, then

c) \( M_W(t) = M_X(t) M_Y(t) \)
\[ = \frac{1}{16} (1 + e^t + e^{2t} + e^{3t})(1 + e^{4t} + e^{8t} + e^{12t}) \]
\[ = \frac{1}{16} \sum_{w=0}^{15} e^{wt} \]
\[ = \sum_{w=0}^{15} e^{wt} \cdot \frac{1}{16} \]

So \( P(W=w) = \frac{1}{16} \) for \( w = 0, 1, 2, 3, \ldots, 15 \).
Let

\[ X_i = \# \text{sick days taken by employee } i, \]

for \( i = 1, 2, 3, 4 \). The total number of sick days taken is

\[ Y = X_1 + X_2 + X_3 + X_4. \]

If the \( X_i \)'s are independent Poisson \( (\lambda = 2) \), then by exercise 5.4-4 (see above), \( Y \) is Poisson \( (\lambda = 8) \). Hence

\[
P(Y > 10) = 1 - P(Y \leq 10)
\]

\[
= 1 - \sum_{y=0}^{10} \frac{e^{-8} \cdot 8^y}{y!}.
\]

\[
= 0.1841.
\]
The population mean is
\[ m = E(X) = \int_{-1}^{1} x \cdot \frac{3}{2} x^2 \, dx = \frac{3}{2} \cdot \frac{1}{4} x^4 \bigg|_{-1}^{1} = 0 \]
and the population variance is
\[ \sigma^2 = E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{3}{2} x^2 \, dx = \frac{3}{2} \cdot \frac{1}{5} x^5 \bigg|_{-1}^{1} = \frac{3}{5} \]

By the CLT, \( \bar{X} = \frac{1}{15} Y \) satisfies
\[ \frac{\bar{X} - 0}{\sqrt{\frac{3/5}{15}}} \approx Z \]
when \( Z \) is standard normal. Thus
\[ \bar{X} = \sqrt{\frac{3/5}{15}} Z = \frac{1}{5} Z \]
is approximately normal with mean 0 and std. dev. \( \frac{1}{5} \), and
\[ Y = 15 \cdot \frac{1}{5} Z = 3Z \]
is approximately normal with mean 0 and std. dev. 3.
\[ P(Y \in [-0.3, 1.5]) = \int_{-0.3}^{1.5} \frac{1}{\sqrt{2\pi} \cdot 3} e^{-\frac{1}{2} \left( \frac{x}{3} \right)^2} \, dx \]

\[ = 0.231290364 \ldots \]

(Note: the error between this and the exact result is about 0.0034.)
By the CLT,

\[
\frac{\bar{X} - 40}{\sqrt{8}/\sqrt{32}} \approx Z
\]

where \( Z \) is std. normal. So,

\[
\bar{X} = \sqrt{\frac{8}{32}} Z + 40
\]

is approximately normal with mean 40 and std. dev.

\[
\sqrt{\frac{8}{32}} = \sqrt{\frac{1}{4}} = \frac{1}{2}
\]

Using the normal PDF and numerical integration,

\[
P(39.75 \leq \bar{X} \leq 41.25)
\]

\[
= \int_{39.75}^{41.25} \frac{1}{\sqrt{2\pi} \cdot \frac{1}{2}} \ e^{-\frac{1}{2} \left( \frac{x-40}{\frac{1}{2}} \right)^2} \, dx
\]

\[
= 0.6853
\]
5.6, #6

1) The population mean is
\[ \mu = \int_{0}^{1} x(1-x) \, dx = \left[ \frac{1}{2}x^2 - \frac{1}{6}x^3 \right]_{0}^{1} = \frac{2}{3} \]

and the population variance is
\[ \sigma^2 = \int_{0}^{1} (x - \frac{2}{3})^2 (1-x) \, dx \]
\[ = \left[ -\frac{x^3}{2} + \frac{5x^2}{3} - \frac{14x}{9} + \frac{4}{9} \right]_{0}^{1} = \frac{2}{9} \]

2) By the CLT, \( \bar{X} \) is approximately normal, with

Mean: \( \frac{2}{3} \) and
Variance: \( \frac{2}{9} \)

So
\[ P\left( \frac{2}{3} \leq \bar{X} \leq \frac{5}{6} \right) = \int_{\frac{2}{3}}^{\frac{5}{6}} \frac{1}{\sqrt{\frac{2}{9}}} e^{-\frac{1}{2} \left( \frac{x - \frac{2}{3}}{\frac{1}{3}} \right)^2} \, dx \]

\[ \approx 0.4332 \]
$E(X) = 24.43, \ Var(X) = 2.20$

$\bar{X} =$ sample mean from random sample of size $N = 30$ from the distribution of $X$.

Then

a) $E(\bar{X}) = E(X) = 24.43$

b) $\text{Var}(\bar{X}) = \frac{\text{Var}(X)}{n} = \frac{2.20}{30} = 0.07333$

c) By the CLT, $\bar{X}$ is approximately normal, with

mean $24.43$ and variance $0.07333$. So

$P(24.17 \leq \bar{X} \leq 24.82)$

$= \int_{24.17}^{24.82} \frac{1}{\sqrt{2\pi(0.07333)}} \exp\left(-\frac{1}{2}\left(\frac{x - 24.43}{0.07333}\right)^2\right) \, dx$

$\approx 0.7566$
The population distribution has mean 2000 and std. dev. 500.

\( \bar{X} \) is sample mean from a random sample of size 25.

By the CLT, \( \bar{X} \) is approximately normal, with mean 2000 and std. dev \( \frac{500}{\sqrt{25}} = 100 \).

So,

\[
P(\bar{X} > 2050) = \int_{2050}^{\infty} \frac{1}{\sqrt{2\pi} \cdot 100} \exp\left(-\frac{1}{2} \left(\frac{x-2000}{100}\right)^2\right) \, dx
\]

\[= 0.3085\]
Let
\[ X_i = \text{# sick days taken by employee } i, \quad 1 \leq i \leq 20. \]

Then \( E(X_i) = 10, \) \( \text{StDev}(X_i) = 2, \) and we are assuming that the \( X_i \)'s are IID.

The total number of sick days taken by all employees is
\[ Y = \sum_{i=1}^{20} X_i \]

We need to find \( m > 0 \) such that
\[ P(Y > m) = 0.20 \]

By the CLT,
\[ Z = \frac{Y - 10}{\frac{2}{\sqrt{20}}} \]

is approximately standard normal. Now
\[ P(Y > m) = P(Z > \frac{m - 10}{\frac{2}{\sqrt{20}}}) \]
So we want
\[ P(Z > \frac{m - 10}{\frac{2}{\sqrt{20}}} ) = 0.20 \]

or equivalently
\[ P(Z \leq \frac{m - 10}{\frac{2}{\sqrt{20}}} ) = 0.80 \]

This says that
\[ \frac{m - 10}{\frac{2}{\sqrt{20}}} = \Phi_{0.80} \]

where \( \Phi_{0.80} \) is the 80th percentile of the standard normal distribution.

Using tables or software, we find \( \Phi_{0.80} = 0.8416 \)

Thus we need
\[ \frac{m - 10}{\frac{2}{\sqrt{20}}} = 0.8416 \]

Solving for \( m \) yields
\[ m = 207.53 \]