Notes on Instantaneous Frequency

Marcus Pendergrass

October 29, 2014

Abstract

Modulation of the phase or frequency of a periodic waveform is a well-known technique for sound synthesis. Usually the modulated waveform is a sinusoid, but other types of waveforms can be modulated as well (e.g. a square wave). But in these cases the classical definition of instantaneous frequency (IF) as the derivative of complex phase may not be appropriate. For instance, when modulating a square wave, one would like to think that the IF of the unmodulated waveform would be constant. This is not the case with the classical definition of IF.

Here we introduce the concept of instantaneous frequency with respect to a periodic waveform. In our setting, the IF of any unmodulated periodic waveform with respect to itself is constant. We go on to explore the basic properties of this notion of IF. We use this concept to rigorously define a time-varying spectrum with respect to IF. This IF spectrum is analogous to the power spectrum of classical Fourier analysis, in that it is based on a certain orthogonal decomposition of a function space.

1 Instantaneous Frequency With Respect to a Periodic Waveform

Let \( r : [0, \infty) \to \mathbb{R} \). Think of \( r(x) \) as the sound value recorded at position \( x \) in some linear recording medium. Let \( p : [0, \infty) \to [0, \infty) \) satisfy \( p(0) = 0 \). Think of \( x = p(t) \) as the position at time \( t \) of a playback head moving through the recording medium. The playback waveform or modulated waveform is then

\[
w(t) = r(p(t)), \quad t \geq 0.
\]

Henceforth we assume that \( r \) is periodic with period 1, and is mean-zero. Think for a moment that \( p \) is strictly increasing: then \( p(t + \Delta t) - p(t) \) is the number of cycles of \( r \) traversed in the time interval \([t, t + \Delta t]\), and hence

\[
\frac{p(t + \Delta t) - p(t)}{\Delta t}
\]

is the average frequency of \( w \) on the interval \([t, t + \Delta t]\). Thus we define the instantaneous frequency (IF) of \( w \) with respect to \( r \) at time \( t \) by

\[
F_{w|r}(t) = p'(t)
\]
The definition makes sense regardless of whether $p$ is increasing or not. If $p$ decreases, then the playback head is moving backwards through the recording medium, and the instantaneous frequency is negative. In any case, the units of $F_{w|r}$ are Hertz $(\text{sec}^{-1})$. If $w$ and $r$ are clear from context (as they are now), we will simply write $F$ for the IF function.

**Remark 1.1.** This differs from the standard definition of instantaneous frequency as the derivative of complex phase. For one thing, in our setting the IF of $w(t) = r(f_0 t)$ with respect to $r$ is clearly $F(t) = f_0$. This is not the case with the standard definition.

**Remark 1.2.** Instantaneous frequency is clearly referenced to the underlying periodic function $r$ in this view. Suppose $w = r \circ p$ where $r = s \circ q$, where both $r$ and $s$ are periodic with period 1. Then $w = s \circ q \circ p$, and we quickly find that

$$F_{w|s}(t) = F_{w|r}(t) \cdot F_{r|s}(p(t))$$

This is the *chain rule* for instantaneous frequency.

## 2 Periodic Modulation

We are particularly interested in IF functions of the form

$$F(t) = f_0 + m(t)$$

Here $f_0$ is the positive carrier frequency, and $m$ is the frequency modulation function. In this case we have

$$p(t) = f_0 t + \int_0^t m(s) \, ds$$

and

$$w(t) = r \left( f_0 t + \int_0^t m(s) \, ds \right)$$

We are interested in conditions under which the modulated waveform $w$ is periodic.

**Theorem 2.1** (Periodic modulation). Suppose $m$ is periodic with period $T_m$, and that $T_m/T_0$ is rational, where $T_0 = 1/f_0$. Suppose also that $m$ is mean-zero over one period. Then the modulated waveform $w$ is periodic with period $T = \text{lcm}(T_0, T_m)$.

**Proof.** Write $T_m/T_0 = p/q$, where $p$ and $q$ are relatively prime integers, and let $T = \text{lcm}(T_0, T_m)$. Note that $T = qT_m = pT_0$. Then

$$w(t + T) = r \left( f_0 t + \int_0^t m(s) \, ds + f_0 T + \int_t^{t+T} m(s) \, ds \right)$$

Now the interval from $t$ to $t+T$ comprises $q$ cycles of $m$. Thus, since $m$ is mean-zero, $\int_t^{t+T} m(s) \, ds = 0$. Also $f_0 T = p$, an integer. Hence

$$w(t + T) = r \left( f_0 t + \int_0^t m(s) \, ds + p \right)$$

$$= r \left( f_0 t + \int_0^t m(s) \, ds \right) \quad \text{(since } r \text{ has period 1)}$$

$$= w(t)$$

2
Remark 2.2. \( r \) need not be mean-zero in Theorem 2.1.

Remark 2.3. Though \( w \) is periodic with period \( T \), its IF function \( F \) generally has a shorter period \( T_m \), shorter by an integral factor (\( T = qT_m \) for some \( q \in \mathbb{N} \)).

Remark 2.4. \( w \) may not be mean-zero. However, one can easily show that, in the scenario of Theorem 2.1, if \( r \) is mean-zero then we have

\[ \int_0^T w(t)F(t) \, dt = 0. \]

Thus the values of \( w \), when weighted by their instantaneous frequency, are mean-zero. Equivalently, \( w \) is orthogonal to its IF function. But more can be said. The waveform \( w \) was produced from \( r \) by varying the playback rate. The integral above models sampling \( w \) at a variable rate \( F \) that matches the playback rate. This gets us back to \( r \), which is mean-zero.

2.1 IF Spectrum

In this section we work under the assumptions of Theorem 2.1. That is, we are working with a fixed IF function of the form

\[ F(t) = f_0 + m(t) \]

where \( m \) is periodic and mean-zero, and where the period \( T_m \) of \( m \) satisfies \( T_m/T_0 = p/q \), where \( T_0 = 1/f_0 \) and \( p \) and \( q \) are relatively prime integers. Consider the vector space

\[ V = \{ r : [0, \infty) \to \mathbb{R} : r \text{ is periodic with period 1 and mean-zero} \} \]

Any \( r \in V \) can be expanded in a Fourier series

\[ r(x) = \sum_{k=1}^{\infty} a_k \cos(2\pi kx) + b_k \sin(2\pi kx), \quad (1) \]

which we think of as the expansion of \( r \) with respect to the orthogonal Fourier basis

\[ \beta = \{ \cos(2\pi kx), \sin(2\pi kx) : k \in \mathbb{N} \} \]

in \( V \). (Orthogonal with respect to the standard inner product on \( V \).) The modulated waveform \( w \) induced by \( F \) and \( r \) can be written

\[ w(t) = r(p(t)) = \sum_{k=1}^{\infty} a_k \cos(2\pi kp(t)) + b_k \sin(2\pi kp(t)). \quad (2) \]

where \( p(t) = \int_0^t F(s) \, ds \). By Theorem 2.1 \( w \) is periodic with period \( T = pT_0 = qT_m \). Note that \( T \) depends only on the fixed IF function \( F \); the image of every \( r \in V \) has period \( T \). It is easy to see that the mapping \( r \mapsto w \) is linear.
The expansion (2) is not a Fourier series, but it is an orthogonal expansion in a certain vector space. To see this let

$$\beta_w = \{\cos(2\pi kp(t)), \sin(2\pi kp(t)) : k \in \mathbb{N}\}$$

and let

$$W = \text{span}\beta_w.$$ 

Consider the bilinear map $\langle w_1w_2\rangle_w$ on $W$ defined by

$$\langle w_1w_2\rangle_w = \int_0^T w_1(t)w_2(t)F(t)\,dt$$

Provided that the IF function $F$ is positive, this defines an inner product on $W$.

**Theorem 2.5.** Let $F$ be positive. Then the set $\beta_w$ is an orthogonal basis for $(W, \langle \cdot, \cdot \rangle_w)$. Moreover the coordinate transformation for the linear map $r \mapsto w$ from $V$ to $W$ with respect to the bases $\beta$ and $\beta_w$ is the identity mapping.

**Proof.** Consider the case $\langle \cos(2\pi kp(\cdot))\sin(2\pi jp(\cdot))\rangle_w$:

$$\langle \cos(2\pi kp(\cdot))\sin(2\pi jp(\cdot))\rangle_w = \int_0^T \cos(2\pi kp(t))\sin(2\pi jp(t))F(t)\,dt$$

Make the substitution $u = p(t)$, $du = F(t)\,dt$. When $t = 0$ we have $u = 0$, and when $t = T$ we have

$$u = \int_0^T f_0 + m(s)\,ds$$

$$= f_0pT_0 + \int_0^{T_0} m(s)\,ds$$

$$= p$$

Hence the transformed integral is

$$\int_0^p \cos(2\pi ku)\sin(2\pi ju)\,du = 0$$

and thus we have orthogonality. The same argument holds in the other two cases. The statement about the coordinate transformation follows from (2).

**Remark 2.6.** All the elements of $\beta_w$ have length $\sqrt{p/2}$.

**Remark 2.7.** In this context we can recover the reference waveform $r$ from $w$: compute the coefficients of $w$ with respect to $\beta_w$; by (1) and (2) these are precisely the coefficients of $r$ with respect to the Fourier basis $\beta$. As a consequence, if the map $F$ is positive, then the map $r \mapsto w$ from $V$ to $W$ is one-to-one and onto.
Refer to (2) as the IF expansion of $w$. In that expansion, denote

$$w_k(t) = a_k \cos(2\pi kp(t)) + b_k \sin(2\pi kp(t))$$

for $k = 1, 2, \cdots$. Then $w = \sum w_k$, and by orthogonality, $\|w\|_{l^p}^2 = \sum \|w_k\|_{l^p}^2$. Note that

$$\|w_k\|_{l^p}^2 = \frac{p}{2}(a_k^2 + b_k^2)$$

Moreover, the IF of $w_k$ with respect to $\cos(2\pi \cdot)$ is $kF(t) = kf_0 + km(t)$. Define the IF spectrum of $w$ as the sequence

$$\{(kF(t), \|w_k\|_{l^p}^2) : k \in \mathbb{N}\}$$

The IF spectrum is analogous to a power spectrum, except now the frequencies are time-varying. (They remain in arithmetic ratio at all times, however.) The magnitudes are not time-varying; in fact, the magnitudes are proportional to the ordinary power spectrum of $r$. Let $r_k$ be the $k^{\text{th}}$ harmonic component of $r$:

$$r_k(t) = a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$$

Then $\|w_k\|_{l^p}^2 = p\|r_k\|^2$. Hence, the IF spectrum of $w$ as defined here is essentially the power spectrum of $r$ “set in motion”: the position at time $t$ of the $k^{\text{th}}$ harmonic is $kF(t)$. 