Math 421: Probability and Statistics I
Note Set 1

Marcus Pendergrass

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1 Course Introduction

Approach of the course: introduce ideas intuitively at first, then develop the rigorous mathematics. "Knowing a concept" means being able to understand and solve problems with it.

1.1 Course Content

1. What is probability?
2. Combinatorics: how to count without counting.
3. Discrete Probability
    (a) Finite and countable sample spaces.
    (b) Random samples and equally likely sample spaces.
    (c) Sample Spaces that aren't equally likely.
    (d) Axioms for discrete probability.
    (e) Inclusion-Exclusion
    (f) Random variables.
    (g) Expected value.
    (h) Conditional probability and Bayes' theorem.
    (i) Independence.
    (j) Jointly distributed random variables.
    (k) Some important discrete probability models.
        i. Discrete uniform distribution.
        ii. The binomial model.
        iii. The multinomial model.
        iv. Hypergeometric model.
        v. Benford's distribution.

(l) Generating functions

4. Continuous Probability

    (a) Continuous sample spaces.
    (b) General probability axioms.
    (c) Random variables, expected value, conditional probability, Bayes' theorem, independence, joint distributions.
    (d) Important continuous probability models.
        i. Continuous uniform distribution.
        ii. Exponential distribution.
        iii. Normal distribution.
        iv. Gamma distribution.
    (e) Generating functions.

5. Limit Theorems

    (a) Weak law of large numbers.
    (b) Strong law of large numbers.
    (c) Central limit theorem.

6. Markov Chains[1]

    (a) Transition probabilities.
    (b) Stationary measures.

2 What is probability?

Probability theory is concerned with meaningful and consistent ways of assigning "likelihoods" – or *probabilities* – to events. These probabilities are interpreted as the long-term frequency of occurrence of the events in question (the *frequentist interpretation*), or the degree of subjective belief an individual has regarding the event (the *epistmic interpretation*). There are other interpretations as well. While there are various interpretations of probability, the mathematical theory of probability is purely axiomatic, hence unambiguous, and independent of any specific interpretive position.

Exercise 2.1. In each case, what interpretation of probability is being used?

1. Toss a fair coin. The probability of "heads" is 0.5.

2. It probably won't snow in Farmville on New Year's Day.

[1]Time permitting.

3 Combinatorics

Combinatorics is the mathematics of counting. It is often useful in probability.

3.1 Multiplication Principle

Example 3.1.

1. A certain state has specified that all license plates must contain exactly 7 characters. The characters must all be capital letters. How many distinct license plates are there?

2. A certain state has specified that all license plates must contain exactly 7 characters. The first three characters must be capital letters, while the last four characters must be digits. How many distinct license plates are there?

3. There are four routes from Town A to Town B, two routes from Town B to Town C, and five routes from Town C to Town D. In how many ways can you drive from Town A to B, C, and then D?

4. How many four-digit numbers are there in which no digit appears more than once? What if leading zeros are not allowed?

Multiplication Principle. If there are m outcomes for experiment 1, and, for each outcome of experiment 1, there are n possible outcomes for experiment 2, then together there are mn outcomes for the two experiments.

Exercise 3.1.

1. A certain state has specified that all license plates must contain exactly 7 characters. The characters must all be either capital letters or digits. How many distinct license plates are there?

2. A certain state has specified that all license plates must contain exactly 7 characters. The characters must all be either capital letters or digits, but no character may appear twice. How many distinct license plates are there?

3. Roll a six-sided die four time, and record the outcomes in order. How many possible outcomes are there?

4. Alice, Bob, Cathy, and Darren have formed a band consisting of 4 instruments. If each member can play all four instruments, how many different arrangements are there? What if Alice and Bob can play all four instruments, while Cathy and Darren can each play only piano and bass?

3.2 Permutations

Example 3.2.

1. How many different batting orders are there for the 9 players on a baseball team?

2. James has 5 books for his literature class, 4 books for his philosophy class, and 2 books for his mathematics class. He wants to arrange them on a bookshelf so that books for the same subject are together, with the literature books first, the philosophy books next, and the mathematics books last. How many arrangements are there?

3. In the previous problem, what if there order of the subjects didn't matter? How many different arrangements are there in this case?

4. How many different arrangements of the letters in the word "peppers" are there?

Factorials. $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. (By convention, $0! = 1$.)

Useful Property. $(n+1)\cdot n! = (n+1)!$

Permutations. A linear arrangement of n distinct objects is called a *permutation.* Note: order matters in a permutation! By the multiplication principle there are $n!$ permutations of n distinct objects. A linear arrangement of r distinct objects chosen from among n possible distinct objects is called a *permutation of n things taken r at a time.* By the multiplication principle the number of permutations of n objects taken r at a time is

$$P_{n,r} = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Another common notation for permutations is $(n)_r$, read "n down r". It means exactly what it says: starting at n, count down r times, multiplying along the way. So $(n)_r = P_{n,r} = n(n-1)(n-2)\cdots(n-r+1) = n!/(n-r)!$.

Exercise 3.2.

1. A club is to choose a president, vice-president, treasurer, and secretary from among its 20 members. How many possible selections are there in all?

2. Three boys and three girls are to sit in a row. How many arrangements are possible? If no two people of the same sex are allowed to sit together, how many arrangements are possible?

3. Six people are to be seated at a round table. Call two such seatings equivalent if the second is simply a rotation of the first. How many different arrangements are there?

4. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white,and 1 is blue. If the child puts the blocks in a row, how many different arrangements are possible?

3.3 Combinations

Example 3.3.

1. At a football game, 3 of the 11 players on the home team are to be chosen to witness the flip of the coin that determines which team receives the initial kick-off. How many such selections are possible?

2. From a group of 5 men and 7 women, how many committees can be chosen consisting of 2 men and 3 women?

3. What is the coefficient of x^3y^5 in the expansion of $(x+y)^8$?

4. You have 8 red marbles and 5 blue marbles. How many ways can you line them up so that no two blue marbles are consecutive?

Binomial Coefficients. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Combinations. A subset of size r from a set of size n is called a *combination of n things taken r at a time.* Note: order does not matter in a combination! By the multiplication principle the number of permutations of n objects taken r at a time is

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Useful Properties.

1. Complement Property: $\binom{n}{r} = \binom{n}{n-r}$

2. Pascal's Identity: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

3. Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Exercise 3.3.

1. How many different five-card hands are there in an ordinary 52-card deck?

2. A certain state has specified that all license plates must contain exactly 7 characters. Three characters must be capital letters, while the other four characters must be digits. How many distinct license plates are there? Repeat, now assuming in addition that no letter nor digit may be repeated.

3. What is the coefficient of x^2y^3 in $(3x+2y)^5$?

4. Consider an 8-by-8 chessboard. Suppose you are constrained to move only to the right and up on the board. Starting at the lower left square, you make your way to the upper right square. How many possible paths are there?

3.4 Multinomial Coefficients

Example 3.4.

1. The Faculty Affairs Committee at a certain school consists of 12 members. These members are divided into three subcommittees: the Subcommittee on Appointments, which has 3 members, the Subcommittee on Policy, which has 5 members, and the Grievance Subcommittee, which has 4 members. How many different selections are possible?

2. What is the coefficient of x^2yz^3 in $(x+y+z)^6$

3. How many ways can 8 chess players be paired up to play four chess games?

Multinomial Coefficients. The number of ways in which n distinct objects can be divided into k distinct groups of sizes n_1, $n_2, \cdots, n_k$ is

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$$

Multinomial Theorem.

$$(x_1 + x_2 + \cdots x_k)^n = \sum_{n_1+n_2+\cdots n_k=n} \binom{n}{n_1, n_2, \cdots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where the sum is over all vectors $(n_1, n_2, \cdots, n_k)$ of nonnegative integers that sum to n.

Exercise 3.4.

1. A dozen individually decorated Easter eggs are to be divided among three people. If each person gets the same number of eggs, how many such divisions are possible?

2. What is the coefficient of $x^3y^2z^2$ in $(2x+3y+cz)^7$?

3. The third-grade class at a certain school consists of 100 students. They are to be loaded onto four identical buses, each of which carries 25 students. How many different arrangements are possible?