Math 421: Probability and Statistics I
Note Set 2

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4 Discrete Probability

Discrete probability is concerned with situations in which you can essentially list all the possible outcomes.

4.1 Finite and countable sample spaces

Example 4.1.

1. What are the possible outcomes when 2 fair dice are rolled?
2. Flip a fair coin, and record the trial number of the first head. What are the possible outcomes?
3. Throw a dart at a dartboard, and measure the distance from the dart to the bullseye. What are the possible outcomes?
4. What will the headline of the *New York Times* be on your 50^{th} birthday? List the possibilities.

Definition 4.2 (Random experiment, sample space)**.** A *random experiment* is a phenomenon whose outcome cannot be predicted ahead of time, but which has a well-defined set of all possible outcomes. The set of all possible outcomes for the random experiment is called its *sample space*.

Sets. Mathematically, a *set* is a collection of distinct objects. The *cardinality* of a set is the number of elements it contains. The cardinality of a set E is denoted by $|E|$. A set is said to be *finite* if its cardinality is a natural number; otherwise, the set is *infinite*. If E is an infinite set, and if there exists a function $f : E \to \mathbb{N}$ that is one-to-one and onto, then we say that the set is *countable*. If a sample space S is either finite or countable, we say that S is a *discrete* sample space.

Remark. Georg Cantor (1845 - 1918) showed that there are infinite sets that are not countable. For example, any open interval in the real line is not countable.

Definition 4.3 (Outcomes and events)**.** Let S be the sample space for a random experiment. An individual member of S is called an *outcome* of the experiment. Any collection of outcomes of S (i.e. a *subset* of S) is said to be an *event*. Suppose $x_0 \in S$ is the actual outcome of the random experiment. If E is an event, and $x_0 \in E$, then we say that the event E has *occurred*. If $x_0 \notin E$, then we say the event E did not occur.

Exercise 4.1.

1. A young couple plans to have three children. In terms of the sequence of boys and girls they have, what are the possibilities?

2. Three balls are drawn randomly from an urn containing 6 white balls and 5 black balls. How many possible outcomes are there?

3. Alfred, Bryan, and Carter take turns flipping a coin (in that order). The first one to get a head wins. What is the sample space?

4. A system is composed of 5 components, each of which is either working or failed. At a random point in time, the system is observed, and the status of each component is ascertained. Describe the sample space. How many outcomes does it have?

4.2 Basic notions of set theory.

Set theory is the basic language of probability.

Definition 4.4 (Subset)**.** A set A is said to be a *subset* of another set B if every member of A is also a member of B. If A is a subset of B, we write $A \subseteq B$.

Definition 4.5 (Complement)**.** Suppose A is an event in a sample space S. (That is, $A \subseteq S$.) The *complement* of A is the set of all elements of S that are not in A. The complement of A is denoted by A^c.

Definition 4.6 (Union)**.** The *union* of two sets A and B consists of all elements that are either in A or in B (or both). The union of A and B is denoted by $A \cup B$.

$$A \cup B = \{x \in S : x \in A \text{ or } x \in B\}$$

Definition 4.7 (Intersection)**.** The *intersection* of two sets A and B consists of all elements that are both in A and in B. The intersection of A and B is denoted by $A \cap B$. Sometimes we shorten this, and simply write AB for the intersection of A and B.

$$A \cap B = \{x \in S : x \in A \text{ and } x \in B\}$$

Note: These concepts extend to unions and intersections of more than two sets.

Definition 4.8 (Empty set)**.** The *empty set* is the set that contains no elements. The empty set is denoted by $\emptyset$.

Definition 4.9 (Equality of sets)**.** Sets A and B are said to be equal if both $A \subseteq B$ and $B \subseteq A$.

Note: If $A \subseteq B$, then every member of A is also a member of B. If $B \subseteq A$, then every member of B is also a member of A. Thus, if both $A \subseteq B$ and $B \subseteq A$, then A and B have the same members.

Theorem 4.10 (Algebra of sets)**.** *Let A, B, and C be subsets of S.*

1. $\emptyset \subseteq A$
2. $S^c = \emptyset$
3. $(A^c)^c = A$
4. $A \cup A^c = S$
5. $A \cap A^c = \emptyset$
6. $A \cup B = B \cup A$ *(commutivity of unions)*
7. $A \cap B = B \cap A$ *(commutivity of intersections)*
8. $(A \cup B) \cup C = A \cup (B \cup C)$ *(associativity of unions)*
9. $(A \cap B) \cap C = A \cap (B \cap C)$ *(associativity of intersections)*
10. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *(intersection distributes over a union)*
11. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *(union distributes over an intersection)*
12. $(A \cup B)^c = A^c \cap B^c$ *(DeMorgan's law for unions)*
13. $(A \cap B)^c = A^c \cup B^c$ *(DeMorgan's law for intersections)*

Definition 4.11 (Power set)**.** Let S be any set. The *power set* of S, denoted by 2^S, is the set of all subsets of S:

$$2^S = \{A : A \subseteq S\}$$

Exercise 4.2.

1. Roll a pair of fair dice, and record the number that appears on each. Describe the sample space. (Use set notation.)
2. Roll a pair of fair dice, and record the number that appears on each. Suppose the outcome is $(2, 4)$. Which of the following events have occurred?

   (a) $A = \{(2,4)\}$
   (b) $B = \{\text{the sum of the dice is even}\}$
   (c) $C = \{(i,j) : \min(i,j) > 2\}$
   (d) $\emptyset$
   (e) S

3. Toss a fair coin repeatedly. Let H_n be the event that a head occurs on toss n, for $n \in \mathbb{N}$. Express each of the following events in terms of the events H_n:

   (a) At least one of the first 4 tosses is a head.
   (b) Exactly one of the first 4 tosses is a head.
   (c) The first head appears on toss number 4.

4. Let S be a finite set with cardinality n. Show that the power set of S has cardinality 2^n.

4.3 Equally likely sample spaces.

Example 4.12.

1. Roll a pair of fair dice. What is the probability that the sum of the two dice is equal to 9?

2. A club has 5 men and 7 women. Pick 5 members of the club "at random." What is the probability that exactly 2 of the chosen people are men?

3. Pick a four-digit number "at random". What is the probability that it has no repeated digits?

4. You have 8 red marbles and 5 blue marbles. Line them up randomly in a row. What is the probability that no two blue marbles appear consecutively?

Equally likely sample spaces. Let S be a finite sample space. If all the outcomes in S have the same probability of occurrence, we say that S is an *equally likely* sample space.

Probabilities in equally likely sample spaces. Let S be an equally likely sample space, with $|S| = n$. Then the probability of each individual outcome $\omega \in S$ is $1/n$. The probability of an event E is just the sum of the probabilities of the outcomes in E. Hence if E contains k outcomes, we have

$$P(E) = k \cdot \frac{1}{n} = \frac{|E|}{|S|} \tag{1}$$

Exercise 4.3.

1. Roll a pair of fair dice. Let E be the event that the maximum of the two dice is 5. List the events in E, and find the probability of E.

2. A five-card poker hand is called a *full house* if it consists of three cards of one denomination, and two cards of another denomination. In a fair deal, what is the probability of getting a full house?

3. What is the probability that a five-card hand contains at least one ace?

4. A *flush* is a card hand in which all cards are of the same suit. What is the probability that a five-card hand is a flush?

4.4 General Discrete Sample Spaces

Not all discrete sample spaces have equally likely outcomes. The probability mass function keeps track of the probabilities of each outcome in the sample space.

Axioms for discrete probability. Let S be a discrete sample space, and let $p : S \to \mathbb{R}$ be a function satisfying the following:

Discrete Probability Axiom 1: $p(x) \geq 0$ for all $x \in S$, and

Discrete Probability Axiom 2: $\sum_{x \in S} p(x) = 1$.

Then we say that the ordered pair (S, p) is a *discrete probability space.* (In applications we often say *discrete probability model.*) The function p satisfying the axiom above is called the *probability mass function* for the probability space. For $x \in S$, we refer to $p(x)$ as the *probability* of x.

Definition 4.13 (Probability of an event)**.** Let (S, p) be a discrete probability space. For any subset E of S, the *probability of E*, denoted by $P(E)$, is defined by

$$P(E) = \sum_{x \in E} p(x) \tag{2}$$

Notes:

1. The function $P : 2^S \to [0, 1]$ defined by (2) is called the *probability measure* for (S, p).

2. If S is an equally likely sample space, then equation (2) is equivalent to equation (1) above.

Theorem 4.14 (Probability as a set function)**.** *Let (S, p) be a discrete probability space. Then*

1. $P(S) = 1$.

2. $0 \leq P(A) \leq 1$ *for all* $A \subseteq S$.

3. If $A \cap B = \emptyset$, *then* $P(A \cup B) = P(A) + P(B)$.

4. *If $E_1, E_2, \cdots$ is a sequence of mutually exclusive events (that is, $E_i \cap E_j = \emptyset$ whenever $i \neq j$), then*

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

5. $P(A^c) = 1 - P(A)$ *for all* $A \subseteq S$.

6. $P(A \cup B) = P(A) + P(B) - P(AB)$

Exercise 4.4.

1. Suppose S is a discrete sample space that is not finite. (That is, S is a countably infinite sample space.) Show that it is impossible for all the outcomes in S to have the same probability.

2. A die is loaded so that the probability that it lands on k is proportional to $1/k$; that is, the probability mass function p satisifies

$$p(k) = \begin{cases} C \cdot \frac{1}{k} & \text{if } k = 1, 2, \cdots, 6 \\ 0 & \text{otherwise} \end{cases}$$

   where C is a constant.

   (a) Find C.

   (b) Is the die more likely to land on an even number, or an odd number?

3. An urn contains 4 red balls and 6 black balls. Pick three balls from the urn at random, and let X be the number of red balls selected. Find the probability model for X.

4. Flip a fair coin 4 times, and let X be the number of heads obtained. Find the probability model for X.