Applied Mathematics
Math 345

Midterm

November 5, 2012
Dr. Pendergrass

On my honor, I have neither given nor received any aid on this work, nor am I aware of any breach of the Honor Code that I shall not immediately report.

Pledged: ______________________________

Print Name: ______________________________

Solve the following problems. The solution to each problem should be clearly presented, in a logical and coherent fashion. Begin each solution on a new sheet of paper. You can use your class notes, my posted lecture notes, and the solutions to all previous assignments in working these problems. You can also use any standard calculus book. You may use Maple (or another CAS) only on problem 6. No other outside resources are allowed (including on-line resources). This exam is due Monday, November 12, at the beginning of class.

1. (Envelopes) When a musician *accents* a note, he or she will typically make the first part of the note (the "attack") very forceful and loud, while the steady-state part of the note will be at a somewhat reduced volume. The "accent edge" in the figure below models this type of attack.

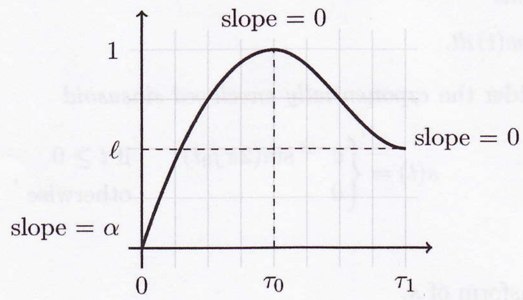


Figure 1. An "accent edge."

Here, α is the slope of the edge at time $t = 0$, τ_0 is the time at which the accent edge is at a maximum, and τ_1 is the end-time for the edge. Use the cubic rising edge model we derived in class to develop a model for the accent edge. Define the accent edge piecewise, using one cubic rising edge to model the first part of the accent edge (up to time τ_0), and then a translated, scaled, and flipped cubic rising edge to model the second part of the accent edge (from time τ_0 to time τ_1). Give a general formula for the accent edge in terms of the rising edge. Give an explicit formula for the accent edge for the case $\tau_0 = 1$, $\alpha = 2$, $\tau_1 = 2$, and $\ell = 0.5$,

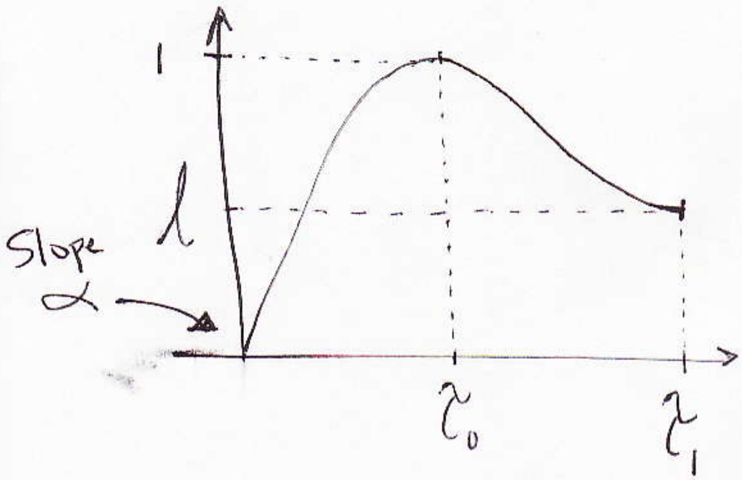
2. (Instantaneous Frequency) Recall that if $s(t) = \cos(2\pi g(t))$, then the *instantaneous frequency function* for s is defined by $f_i(t) = g'(t)$.
   (a) Let $s(t) = \cos(2\pi f_0 t + \frac{W}{2} e^{-at} \sin(2\pi f_m t))$. Find the instantaneous frequency function for s.
   (b) Suppose $s(t) = \cos(2\pi g(t))$, and the instantaneous frequency function for s is $f_i(t) = f_0 + \rho t^n$. Find an explicit formula for $s(t)$.

3. (Pitch Siren) Recall that *pitch* is the base-2 logarithm of frequency. Accordingly, the *instantaneous pitch* of $s(t) = \cos(2\pi g(t))$ is $p_i(t) = \log_2(f_i(t))$, where f_i is the instantaneous frequency function for s. Suppose that the instantaneous pitch function is given by

$$p_i(t) = p_0 + \frac{W}{2} \sin(2\pi f_m t)$$

Find a formula for $s(t)$. (Your formula will contain an integral.)

#1



The "Accent Edge" $a(t)$

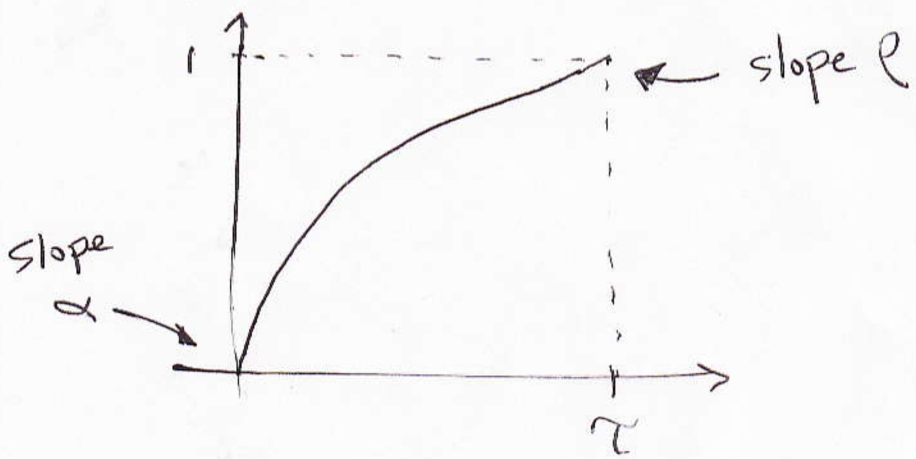
$\alpha = a'(0)$: attack

τ_0 : rise time

τ_1 : end time

l : end level

Recall our cubic rising edge $e(t) = e(t; \tau, \alpha, \rho)$



α = attack

ρ = release

τ = rise time

Note that for $0 \le t \le \tau_0$ we have

$$a(t) = e(t; \tau_0, \alpha, 0)$$

For $\tau_0 \le t \le \tau_1$, the accent edge is a scaled, flipped, and shifted version of the rising edge, namely

$$a(t) = \ell + (1-\ell)\left(1 - e(t-\tau_0;\, \tau_1-\tau_0,\, 0,\, 0)\right)$$

for $t \in [\tau_0, \tau_1]$.

Hence we may write the accent edge in terms of the rising edge as follows

$$a(t) = \begin{cases} e(t;\, \tau_0,\, \alpha,\, 0) & \text{if } 0 \le t \le \tau_0 \\ \ell + (1-\ell)\left(1 - e(t-\tau_0;\, \tau_1-\tau_0,\, 0,\, 0)\right) & \text{if } \tau_0 \le t \le \tau_1 \end{cases}$$

Recall from class that the cubic rising edge is given by

$$e(t) = \alpha t + \frac{(3-(2\alpha+\ell)\tau)}{\tau^2} t^2 + \frac{(\alpha+\ell)\tau - 2}{\tau^3} t^3$$

Using this in the previous expression for $a(t)$ gives

$$a(t) = \begin{cases} \alpha t + \dfrac{3-2\alpha\tau_0}{\tau_0^2} t^2 + \dfrac{\alpha\tau_0 - 2}{\tau_0^3} t^3 & \text{if } 0 \le t \le \tau_0 \\ \ell + (1-\ell)\left(1 - 3\left(\dfrac{t-\tau_0}{\tau_1-\tau_0}\right)^2 + 2\left(\dfrac{t-\tau_0}{\tau_1-\tau_0}\right)^3\right) & \text{if } \tau_0 \le t \le \tau_1 \end{cases}$$

In particular, when $\tau_0 = 1$, $\tau_1 = 2$, $\alpha = 2$, $\ell = \frac{1}{2}$, we get

$$a(t) = \begin{cases} 2t - t^2 & \text{if } 0 \le t \le 1 \\ 1 - \frac{3}{2}(t-1)^2 + (t-1)^3 & \text{if } 1 \le t \le 2 \end{cases}$$

#2

a) $g(t) = f_0 t + \frac{1}{2\pi} \frac{W}{2} e^{-at} \sin(2\pi f_m t)$

So

$$f_i(t) = g'(t)$$
$$= f_0 + \frac{W}{4\pi}\left(-a e^{-at} \sin(2\pi f_m t) + e^{-at} 2\pi f_m \cos(2\pi f_m t)\right)$$

$$= f_0 + \frac{W}{4\pi} e^{-at}\left(2\pi f_m \cos(2\pi f_m t) - a \sin(2\pi f_m t)\right)$$

b) Referenced from $t=0$, we have

$$g(t) = \int_0^t f_i(s)\, ds = \int_0^t f_0 + \rho s^n \, ds$$

$$= f_0 t + \frac{\rho}{n+1} t^{n+1}$$

So in general

$$s(t) = \cos\left(2\pi f_0 t + \frac{2\pi \rho}{n+1} t^{n+1} + \phi\right)$$ "polynomial chirp"

#3

$$f_i(t) = 2^{p_i(t)} = 2^{p_0 + \frac{W}{2}\sin(2\pi f_m t)}$$

So

$$g(t) = \int_0^t f_i(s)\, ds = \int_0^t 2^{p_0 + \frac{W}{2}\sin(2\pi f_m s)}\, ds$$

$$= 2^{p_0} \int_0^t 2^{\frac{W}{2}\sin(2\pi f_m s)}\, ds$$

and thus

$$s(t) = \cos\left(2\pi \cdot 2^{p_0} \int_0^t 2^{\frac{W}{2}\sin(2\pi f_m s)}\, ds + \phi\right)$$

#4

We have $a_k = \frac{1}{k^2}$, $b_k = \frac{1}{k}$, for $k = 1, 2, 3, 4$

The power spectrum is

$$\left\{ \left(kf_0 , \tfrac{1}{2}\left(a_k^2 + b_k^2 \right) \right) : 1 \le k \le 4 \right\}$$

$$= \left\{ \left(kf_0 , \tfrac{1}{2}\left(\tfrac{1}{k^4} + \tfrac{1}{k^2} \right) \right) : 1 \le k \le 4 \right\}$$

$$= \left\{ (f_0, 1), \left(2f_0, \tfrac{5}{32}\right), \left(3f_0, \tfrac{5}{81}\right), \left(4f_0, \tfrac{17}{512}\right) \right\}$$

The phase spectrum is

$$\left\{ \left(kf_0 , \arctan\left(\tfrac{b_k}{a_k}\right) \right) : 1 \le k \le 4 \right\}$$

$$= \left\{ \left(kf_0 , \arctan(k) \right) : 1 \le k \le 4 \right\}$$

$$= \left\{ \left(f_0, \tfrac{\pi}{4}\right), \left(2f_0, \arctan(2)\right), \left(3f_0, \arctan(3)\right), \left(4f_0, \arctan(4)\right) \right\}$$

#5

a) $$\|s\|^2 = \frac{T}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

$$= \frac{T}{2} \sum_{k=1}^{\infty} \left((\alpha^{k-1})^2 + (\beta^{k-1})^2 \right)$$

$$= \frac{T}{2} \sum_{k=0}^{\infty} \left(\alpha^{2k} + \beta^{2k} \right)$$

$$= \frac{T}{2} \left(\frac{1}{1-\alpha^2} + \frac{1}{1-\beta^2} \right)$$

b) $$\langle s, w \rangle = \frac{T}{2} \left(\sum_{k=1}^{\infty} a_k c_k + \sum_{k=1}^{\infty} b_k d_k \right)$$

$$= \frac{T}{2} \left(\sum_{k=1}^{\infty} \alpha^{k-1} \beta^{k-1} + \sum_{k=1}^{\infty} \beta^{k-1} (-\alpha^{k-1}) \right)$$

$$= 0$$

So s and w are orthogonal.

#6

(See accompanying Maple worksheet for parts a) through c).)

d). As the parameter "a" increases, the time-domain function gets more "pulse like", an the frequency-domain representation becomes less "spike-like", and more spread-out in frequency. (It also becomes less energetic as "a" increases, due to it's shortened effective duration.)