Patterns and Schemes for Algorithmic Composition

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1 Introduction

In this paper we study a recursive technique for the algorithmic composition of music. Our method is completely deterministic, and can generate both melodies and accompanying parts. The resulting musical structures that have fractal-like properties of self-similarity on multiple scales. The main ideas in this paper are:

- *Composition of pitch patterns.* Pitch patterns can be thought of as sequences of transpositions, and the operation of composition defined in Section 2 is a musically natural way of applying these transposition sequences repeatedly. In our framework the set of all pitch patterns forms a semigroup under composition, and the semigroup structure suggests a useful perspective for thinking about melodic structure.

- *Substitution schemes.* Substitution schemes generalize the composition operation, while preserving its important structural properties.

These ideas are defined precisely in Section 2, and their basic mathematical properties are drawn out. Several examples illustrating how these ideas can be used to create interesting musical pieces are given in Section 3. The connecting thread between all these examples is the idea of iterating a composition or substitution scheme. In Section 4 we explore techniques for solving substitution schemes, and in the next section we discuss the self-similarity of the solution sequences. The algebraic structure of the semigroup of pitch patterns is further elaborated in Section 6, and the musical implications of factorization in this semigroup are illustrated.

Our methods are related to existing techniques, particularly the work of the American structuralist composer Tom Johnson [cite]. We conclude with a discussion of these relationships, and directions for future work.

2 Patterns and Schemes

We represent musical pitches with integers. For instance, centering at middle A (440 Hertz), we might map the chromatic scale as

$$\ldots, \quad A\flat \longleftrightarrow -1, \quad A \longleftrightarrow 0, \quad A\sharp \longleftrightarrow 1, \quad \ldots$$

Many other scales and mappings are possible, of course. A *pitch pattern* is a finite ordered sequence of pitches. Pitch patterns are represented by vectors with integer entries. We will denote the set of all pitch patterns by $\mathbb{Z}^* = \cup_{n=1}^{\infty} \mathbb{Z}^n$. It will also be useful to represent pitch patterns graphically as in Figure 1, which shows the three-note pitch pattern $(1, -1, 0)$.

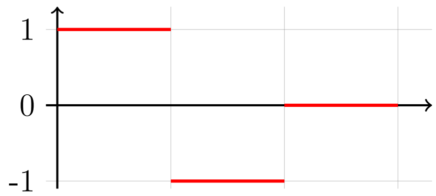

Figure 1. The pitch pattern $\alpha = (1, -1, 0)$.

Concatenation, transposition, and inversion are the basic operations on pitch patterns. If α and β are pitch patterns of length m and n, we denote the concatenation of α followed by β by (α, β):

$$(\alpha, \beta) = (\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1})$$

Transposition of a pitch pattern α by $k \in \mathbb{Z}$ scale steps is denoted by $k + \alpha$. Here the addition of scalar and a vector is component-wise:

$$k + \alpha = (k + \alpha_0,\ k + \alpha_1,\ \ldots,\ k + \alpha_{m-1})$$

Inversion of α about 0 is given by $-\alpha$, while inversion about k is $2k - \alpha$.

Composition of Pitch Patterns. To motivate the definition of composition of pitch patterns, think of a pitch pattern as a sequence of transpositions applied to a single reference pitch, rather than as a sequence of pitches. Now what if the reference pitch isn't a single note, but rather some more complex sound? The natural thing to do is to apply the transpositions specified by the pitch pattern sequentially to that sound.

Definition 1. Let α be a pitch pattern of length m, and β a pitch pattern of length n. The *composition* $\alpha\beta$ is the pitch pattern of length mn given by

$$\alpha\beta = (\alpha_0 + \beta,\ \alpha_1 + \beta,\ \ldots,\ \alpha_{m-1} + \beta)\,.$$

Thus the composition $\alpha\beta$ consists of concatenated copies of β, with each copy transposed by an amount specified by the entries in α. For instance if $\alpha = (1, -1, 0)$ and $\beta = (0, 1)$, then $\alpha\beta = (\underline{1, 2}, \underline{-1, 0}, \underline{0, 1})$.

In general composition is not commutative, but it is associative. (See Section 6 below.) Thus the set of all pitch patterns forms a semigroup under composition. The pitch pattern (0) acts as an identity for this operation, so we actually have a monoid. The way that a pitch pattern factors in this monoid reveals certain structural aspects of the melody. As we've already seen, a pitch pattern $\gamma = \alpha\beta$ consists of concatenated copies of β, with each copy transposed by an amount determined by α. Thus β defines an "inner structure" for γ, while α defines an "outer structure."

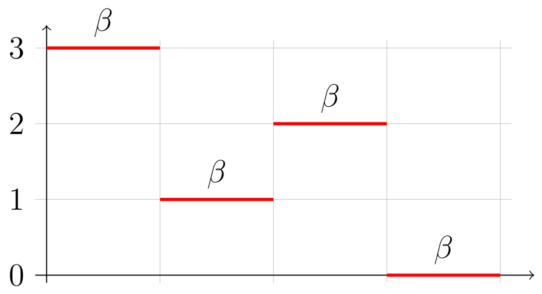

Figure 2. Inner and outer structure for $\gamma = \alpha\beta$. Here $\alpha = (3, 1, 2, 0)$.

Substitution Schemes. In a composition of pitch patterns, each segment of the outer structure contains the same inner structure. We can extend this idea by allowing different patterns to be copied into the segments of the outer structure. Consider Figure 3, which shows a *substitution scheme* for two pitch patterns α and β. Note that the segments of each each pattern have been labelled with either α or β.

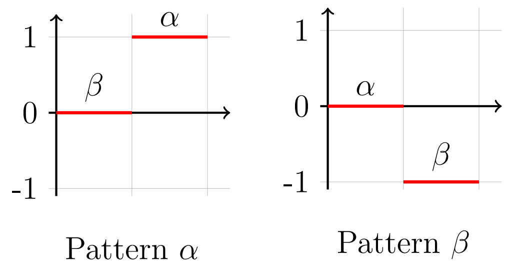

Figure 3. A substitution scheme.

Iterating the substitution scheme means replacing each segment with the appropriately transposed pitch pattern specified by that segment's label. The next figure shows the result of iterating the substitution scheme of Figure 3.

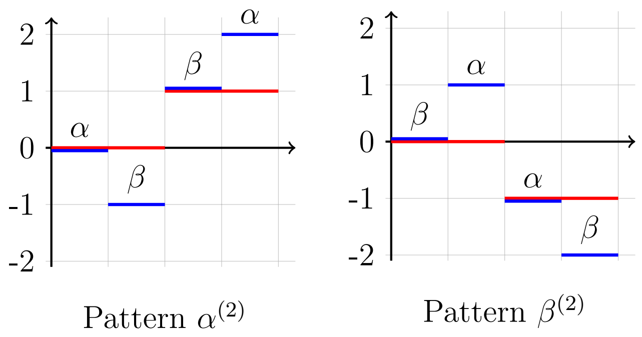

Figure 4. One iteration of the substitution scheme of Figure 3. The iterated patterns are $\alpha^{(2)} = (0, -1, 1, 2)$ and $\beta^{(2)} = (0, 1, -1, -2)$.

Note that the iterated patterns inherit the labels from the substitution scheme, and so can be iterated again. We will denote the n^{th} iterates of the patterns in a substitution scheme by $\alpha^{(n)}$, $\beta^{(n)}$, and so on.

A more compact way to specify a substitution scheme is through a set of equations specifying the iteration process. For instance, using our notations for concatenation and transposition, the system in Figure 3 can be specified by the recursions

$$\begin{aligned} \alpha^{(n+1)} &= \left(\beta^{(n)}, 1 + \alpha^{(n)}\right) \\ \beta^{(n+1)} &= \left(\alpha^{(n)}, -1 + \beta^{(n)}\right) \end{aligned} \tag{1}$$

along with the initial conditions $\alpha^{(0)} = (0)$, $\beta^{(0)} = (0)$. We also allow inversions in a substitution scheme, but remark in passing that any scheme with inversions can be re-written as a (larger) scheme with no inversions.

Example 1. For the substitution scheme (3) in Figure 3, it is clear that $\beta^{(n)} = -\alpha^{(n)}$ for all $n \in \mathbb{N}$. The first few iterates of α are

$$\begin{aligned} \alpha &= (0, 1) \\ \alpha^{(2)} &= (0, -1, 1, 2) \\ \alpha^{(3)} &= (0, 1, -1, -2, 1, 0, 2, 3) \\ \alpha^{(4)} &= (0, -1, 1, 2, -1, 0, 2, 3, 1, 2, 0, -1, 2, 1, 3, 4) \end{aligned}$$

3 Generating Melodies and Accompaniments

The ideas of composition and substitution schemes can be used to create interesting musical pieces in a variety of ways. The first step is to fix a mapping from the set of pitch patterns into some "music space," typically a scale indexed by the integers. Melodies with recognizable structure can then be generated by applying this mapping to pitch patterns generated by composition or substitution schemes. The following two examples scout out some of the possibilities.

Example 2 (Iterating a pitch pattern.)**.** Melodies with easily recognized structure can be created by iterating the map $x \mapsto \alpha x$, where α is some fixed pitch pattern of short length. The iterates are of the form $\alpha^n x$, and thus have the structure defined by α on multiple scales. Lower-order iterates can be "layered" on top of each other to create polyphonic pieces in which the individual parts display the structure of a single iterate.

Consider Figure 5 below, which shows a phrase built from $\alpha = (2, 0, 1)$ and its first two iterates. For this realization, pitch indices were mapped into the C major scale without the fourth (i.e. F), with pitch 0 mapping to middle C. The melody line is given by α^3, which naturally resolves into three groups of three triplets each. The lower-order iterates provide a correlated accompaniment, with the duration of pitches in α^{3-i} equal to 3^i times the duration of pitches in the melody α^3 for $i = 1, 2$. Note that the melody and accompanying parts combine to produce a coherent harmonic movement in the phrase, from A minor to G major. Of course, this is a function of both the original pitch pattern and the chosen mapping of pitch indices to notes.

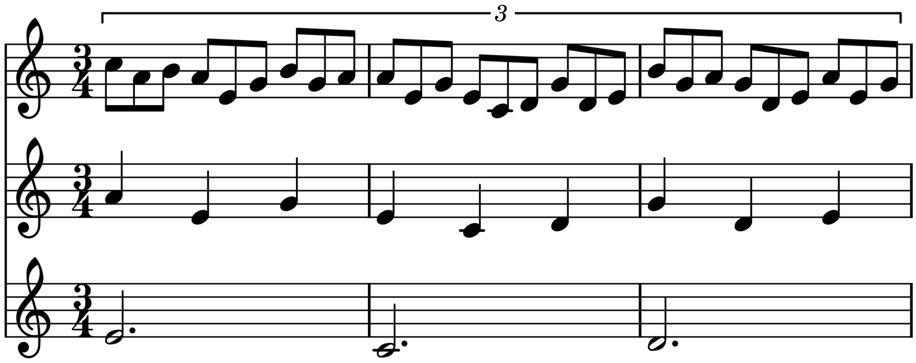

Figure 5. Realization of α^3, with accompaniment provided by α^2 and α.

Example 3 (Iterating a substitution scheme.)**.** Consider again the substitution scheme of Figure 3:

$$\begin{aligned}\alpha^{(n+1)} &= \left(\beta^{(n)}, 1 + \alpha^{(n)}\right)\\ \beta^{(n+1)} &= \left(\alpha^{(n)}, -1 + \beta^{(n)}\right)\end{aligned}$$

with $\alpha^{(0)} = \beta^{(0)} = (0)$. Inductively we see that $\beta^{(n)} = -\alpha^{(n)}$ for all n, so as melodic lines $\alpha^{(n)}$ and $\beta^{(n)}$ are in strict contrary motion. When these lines are played together, we have an example of linear counterpoint. In Figure 6 we show a mapping of $\alpha^{(5)}$ and $\beta^{(5)}$ to the C major scale, Lydian mode. (In other words, pitch index 0 maps to F.)

Figure 6. Counterpoint derived from $\alpha^{(5)}$ and $\beta^{(5)}$.

One could obviously layer these lines above a harmonic accompaniment defined by the lower-order iterates of α and β, as in Example 2. Combining techniques such as these can produce fully realized compositions of remarkable intricacy and beauty.

We remark that complete recordings of these and other compositions produced by the methods of this paper are available in the Online Supplementary Materials.

4 Solving a Substitution Scheme

Solving a substitution scheme means finding a non-recursive formula or algorithm to compute the entries of its iterates. Whether this is tractable or not depends on the complexity of the substitution scheme. The simplest case is iteration of the composition map $x \mapsto \alpha x$ starting from the initial condition $x = (0)$, which results in sequences of the form α^n. In this case there is a simple formula for the solution, given by the following theorem.

Theorem 1. *Let α be a pitch pattern with length $m \geq 2$, and let n be a positive integer. Then for $0 \leq i < m^n$, we have*

$$\alpha_i^n = \sum_{k=0}^{m-1} \alpha_{i(k)}$$

where $i(k)$ is the k^{th} digit in the n-digit base-m expansion of i, including leading zeros.

The proof is a straightforward induction on n. As an example of using the theorem, consider $\alpha = (1, -1, 0, 2)$. To find α_{29}^5, we first find the 5-digit base-4 expansion of $i = 29$, which is 00131_4. Applying the theorem we get $\alpha_{29}^5 = \alpha_0 + \alpha_0 + \alpha_1 + \alpha_3 + \alpha_1 = 2$.

For more general substitution schemes, non-recursive solutions can be more intricate, and more difficult to find. Consider a substitution scheme of the following form:

$$\begin{aligned} \alpha^{(n+1)} &= \left(\alpha^{(n)}, \alpha_1 + \beta^{(n)}\right) \\ \beta^{(n+1)} &= \left(\beta^{(n)}, \beta_1 + \alpha^{(n)}\right) \\ \alpha^{(0)} &= \beta^{(0)} = (0) \end{aligned} \tag{2}$$

Here α_1 and β_1 are fixed integers. Note that this system is *progressive*, in the sense that iterate n of a pattern (either α or β) is the initial subsequence of iterate $n+1$ of that pattern. For instance if $\alpha_1 = 2$ and $\beta_1 = -1$, then the first two iterates are $\alpha^{(1)} = (0, 2)$, $\beta^{(1)} = (0, -1)$, $\alpha^{(2)} = (0, 2, 2, 1)$, and $\beta^{(2)} = (0, -1, -1, 1)$. Progressivity implies that the limiting sequences $\alpha^\infty = \lim_{n\to\infty} \alpha^{(n)}$ and $\beta^\infty = \lim_{n\to\infty} \beta^{(n)}$ both exist.

We can use a generating function approach to solve this system. Define polynomials $p_{n,k}$ and $q_{n,k}$ by

$$\begin{aligned} p_{n,k}(t) &= \sum_i \delta(\alpha_i^{(n)}, k)\, t^i \\ q_{n,k}(t) &= \sum_i \delta(\beta_i^{(n)}, k)\, t^i \end{aligned}$$

Here $\delta(j, k)$ is the Dirac delta function, equal to one if its arguments are equal to each other, and zero otherwise. Thus the polynomials $p_{n,k}$ and $q_{n,k}$ encode in their exponents the positions in which the value k occurs in $\alpha^{(n)}$ and $\beta^{(n)}$. For instance, for $\alpha^{(2)} = (0,2,2,1)$, we have $p_{2,0}(t) = 1$, $p_{2,1}(t) = t^3$, and $p_{2,2}(t) = t + t^2$. (If k does not appear in $\alpha^{(n)}$, we define $p_{n,k}(t)$ to be the zero polynomial, and similarly for $q_{n,k}$.) Now define functions $P_n(s,t)$ and $Q_n(s,t)$ by

$$P_n(s,t) = \sum_{k\in\mathbb{Z}} p_{n,k}(t)e^{iks}$$
$$Q_n(s,t) = \sum_{k\in\mathbb{Z}} q_{n,k}(t)e^{iks}$$

From the substitution scheme we can derive a recurrence for P_n and Q_n:

$$P_{n+1}(s,t) = P_n(s,t) + t^{2^n}e^{i\alpha_1 s}Q_n(s,t)$$
$$Q_{n+1}(s,t) = t^{2^n}e^{i\beta_1 s}P_n(s,t) + Q_n(s,t)$$

with initial conditions

$$P_0(s,t) = Q_0(s,t) = 1$$

Taking into account the initial conditions, we can solve the recursion as a matrix product,

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \prod_{k=0}^{n-1} \begin{pmatrix} 1 & t^{2^k}e^{i\alpha_1 s} \\ t^{2^k}e^{i\beta_1 s} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now in this particular case it turns out that the matrices in this product are simultaneously diagonalizable in k, and this can be exploited to expand the matrix product, as follows. Call the the number of ones in the binary expansion of an integer i its *binary weight*, and let $|i|_2$ denote the binary weight of i. Define $\rho_k(t)$ to be the sum of all powers of t with binary weight k:

$$\rho_k(t) = \sum_{|\ell|_2=k} t^\ell$$

Then the above expressions for P_n and Q_n expand to

$$P_n(s,t) = \sum_{j=0}^{\lfloor n/2\rfloor} \rho_{2j}(t)e^{ij(\alpha_1+\beta_1)s} + \sum_{j=0}^{\lfloor (n-1)/2\rfloor} \rho_{2j+1}(t)e^{i[j(\alpha_1+\beta_1)+\alpha_1]s}$$
$$Q_n(s,t) = \sum_{j=0}^{\lfloor n/2\rfloor} \rho_{2j}(t)e^{ij(\alpha_1+\beta_1)s} + \sum_{j=0}^{\lfloor (n-1)/2\rfloor} \rho_{2j+1}(t)e^{i[j(\alpha_1+\beta_1)+\beta_1]s}$$

Taking limits as n approaches infinity (which exist by progressivity), we get the following theorem, which effectively solves the substitution scheme (2).

Theorem 2. *The limiting sequences α^∞ and β^∞ of the substitution scheme* (2) *are characterized by:*

$$\alpha_i^\infty = \begin{cases} k(\alpha_1+\beta_1) & \text{if } |i|_2 = 2k \\ k(\alpha_1+\beta_1)+\alpha_1 & \text{if } |i|_2 = 2k+1 \end{cases}$$

and

$$\beta_i^\infty = \begin{cases} k(\alpha_1+\beta_1) & \text{if } |i|_2 = 2k \\ k(\alpha_1+\beta_1)+\beta_1 & \text{if } |i|_2 = 2k+1 \end{cases}$$

For a second example, consider the non-progressive system of Example 1,

$$\begin{aligned} \alpha^{(n+1)} &= \big(\beta^{(n)}, 1+\alpha^{(n)}\big) \\ \beta^{(n+1)} &= \big(\alpha^{(n)}, -1+\beta^{(n)}\big) \\ \alpha^{(0)} &= \beta^{(0)} = (0) \end{aligned} \tag{3}$$

We have seen that $\beta^{(n)} = -\alpha^{(n)}$ for all n, so it suffices to find a non-recursive means of calculating $\alpha_i^{(n)}$. Using the same generating function approach, we can show that $\alpha_i^{(n)}$ again depends only on the binary expansion of i, but in a more complicated way. Consider the infinite dihedral group Dih_∞, which is generated by two elements $\{a, b\}$ along with the relations $a^2 = 1$ and $(ab)^2 = 1$. Every finite product in Dih_∞ can be reduced uniquely to the form $b^k a^\ell$ where $k \in \mathbb{Z}$ and ℓ is either 0 or 1. Define a map $f : \mathrm{Dih}_\infty \to \mathbb{Z}$ by

$$f(g) = k \quad \text{if and only if} \quad g = b^k a^\ell \text{ for some } k \in \mathbb{Z},\ \ell \in \{0, 1\} \tag{4}$$

Given a positive integer n, define a map h_n from $\mathbb{Z}_{2^n}$ into Dih_∞ as follows: for $i \in \mathbb{Z}_{2^n}$, let $h_n(i)$ be the word in Dih_∞ obtained by replacing each zero (including leading zeros) in the n-digit binary expansion of i with an a, and each one with a b.

Theorem 3. *For the substitution scheme* (3)*, we have*

$$\alpha_i^{(n)} = f(h_n(i)), \quad 0 \le i < 2^n$$

5 Self Similarity

Sequences obtained by iterating the composition map or a substitution scheme have well-defined levels of structure. The theorems from the previous section can provide insight into this structure.

Consider sequences of the form α^n, which arise from iterating the composition map $x \mapsto \alpha x$. Structurally, the identity $\alpha^n = \alpha^{n-k}\alpha^k$ for all $k = 0, 1, \ldots p$ says that each power α appears in α^n as both an inner and an outer structure. More can be said, however. According to Theorem 1, if α has length $m \ge 2$, then the value of α_i^n is a symmetric function of the base-m digits of i. Thus if the base-m expansion of j is a permutation of the base-m expansion of i, then we have $\alpha_j^n = \alpha_i^n$. Now any given feature of the inner structure of α^n is manifested by entries α_i^n, for i running in some contiguous range, say $i_1 \le i \le i_2$. In turn, the base-m expansions of these indices are characterized by a block B_0 of least-significant digits that is incrementing successively as i runs from i_1 to i_2, and a block B_1 of most-significant digits that is unchanging. Consider now a block-permutation in which B_0 and B_1 are interchanged, but the digits within each block remain in order. Incrementing the digits in B_0 now results in the same values of α_i^n as before, in the same order as before, but these values are now part of the outer structure of α^n. Similarly, a permutation which inserts B_0 somewhere in the middle of the base-m expansion will manifest the original inner structure of α^n as a "mid-level" structure. Even a permutation that merely preserves the order of the digits in B_0, but not their contiguity, will still produce the same series of values as before, in the same order. Thus we see that any feature of structure at any level is manifested at all other levels. In this sense, sequences of the form α^n are like fractals such as the Koch snowflake.

Substitution schemes generalize pitch pattern composition, and their self-similarities can be correspondingly more intricate. For the progressive system (2), Theorem 2 implies that the value of an iterate at a particular index i depends only on the binary weight of i. In this sense, this system is similar to a simple composition map. For the non-progressive system (3) the value of $\alpha_i^{(n)}$ is not determined by the binary weight of i, but depends on actual arrangement of the zeros and ones in the expansion in a complicated way. Moreover, there exist substitution schemes in which the value of a sequence at position i is not determined by a base-m expansion at all. These examples show that substitution schemes can exhibit a wide variety of self-similar structures.

6 Semigroup Structure of $\mathbb{Z}^*$

The principle left ideal of a pitch pattern $\beta \in \mathbb{Z}^*$ is given by $\mathbb{Z}^*\beta = \{s\beta : s \in \mathbb{Z}^*\}$, and consists of all pitch patterns that have inner structure β. If two pitch patterns β and γ have the same principle left ideals, we say they are *left-related*, and write $\beta\mathcal{L}\gamma$. Left-related pitch patterns generate the same set of melodies via composition on the left. Left-relatedness is an equivalence relation. Similarly, the principle right ideal $\alpha\mathbb{Z}^* = \{\alpha s : s \in \mathbb{Z}^*\}$ consists of all pitch patterns that have the same outer structure α. If α and γ share the same principle right ideal, they are said to be *right related*, and we write $\alpha\mathcal{R}\beta$. Finally, the principle two-sided ideal $\mathbb{Z}^*\gamma\mathbb{Z}^* = \{s_1\gamma s_2 : s_1, s_2 \in \mathbb{Z}^*\}$ consists of all melodies that share the same "mid-level" structure γ, and induces an equivalence relation $\mathcal{J}$.

The equivalence relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ are the most basic of *Green's relations* [cite], which give information about the divisibility structure of a semigroup. Additionally, say that α and β are *transpositionally related* if there exists $k \in \mathbb{Z}$ such that $\beta = (k)\alpha$, and denote this by $\alpha\mathcal{T}\beta$. It is not difficult to show that for the semigroup $\mathbb{Z}^*$ of pitch patterns, we have $\mathcal{L} = \mathcal{R} = \mathcal{J} = \mathcal{T}$. (In fact, all of Green's relations reduce to $\mathcal{T}$ on $\mathbb{Z}^*$.) The musical interpretation is that if one pitch pattern is a transposition of another, then the two pitch patterns are equivalent in terms of the melodies they can generate through the operation of pitch pattern composition. As a consequence, there is no loss in the variety of melodies that can be generated when we restrict the composition operation to $\mathbb{Z}_0^* \times \mathbb{Z}_0^*$, where $\mathbb{Z}_0^*$ is the subset of pitch patterns whose first entry is 0.

The basic facts about the operations of composition, concatenation, and inversion are collected in Theorem 4

Theorem 4.

1. *The set of pitch patterns $\mathbb{Z}^*$ forms a monoid under composition, with identity* (0)*.*

2. *For the monoid $\mathbb{Z}^*$, we have $\mathcal{L} = \mathcal{R} = \mathcal{J} = \mathcal{T}$.*

3. Transpositions commute with all elements of $\mathbb{Z}^*$.

4. For all α *and* β *in* $\mathbb{Z}^*$,

$$-(\alpha\beta) = (-\alpha)(-\beta)$$

and

$$(\alpha, \beta)\gamma = (\alpha\gamma, \beta\gamma)$$

6.1 Factoring Melodies

Coming soon!

7 Proofs of Main Results

Proof of Theorem 1. First note that if α and β are pitch patterns of length m and n respectively, and i is between 0 and $mn-1$ inclusive, then the i^{th} entry of the composition $\alpha\beta$ is

$$(\alpha\beta)_i = \alpha_q + \beta_r$$

where q and r are the quotient and remainder when i is divided by n. This follows directly from the definition of composition.

To prove the theorem, assume that α has length $m \geq 2$. For $0 \leq i < m$, call α_i the *value* of i, considered as a base-m digit. We induct on the exponent in α^n. The theorem is trivially true for $n = 1$. Assume the theorem holds for some positive integer n, and consider α^{n+1}. We have

$$\alpha_i^{n+1} = (\alpha^n\alpha)_i = \alpha_q^n + \alpha_r$$

where q and r are the quotient and remainder when i is divided by m. Now r is the lowest-order digit in the base-m expansion of i, and q contains the rest of the digits. By the inductive assumption α_q^n is the sum of the values of the base-m digits of q, and therefore α_i^{n+1} is the sum of the values of the base-m digits of i, completing the induction.

Proof of Theorem 2. Define polynomials $p_{n,k}$ and $q_{n,k}$ by

$$p_{n,k}(t) = \sum_i \delta(\alpha_i^{(n)}, k)\, t^i$$
$$q_{n,k}(t) = \sum_i \delta(\beta_i^{(n)}, k)\, t^i$$

Here $\delta(j,k)$ is the Dirac delta function, equal to one if its arguments are equal to each other, and zero otherwise. The polynomials $p_{n,k}$ and $q_{n,k}$ encode in their exponents the positions in which the value k occurs in $\alpha^{(n)}$ and $\beta^{(n)}$. For instance, for $\alpha^{(2)} = (0, 2, 2, 1)$, we have $p_{2,0}(t) = 1$, $p_{2,1}(t) = t^3$, and $p_{2,2}(t) = t + t^2$. (If k does not appear in $\alpha^{(n)}$, we define $p_{n,k}(t)$ to be the zero polynomial, and similarly for $q_{n,k}$.)

From the substitution scheme we immediately get the following linear recursions for $p_{n,k}$ and $q_{n,k}$,

$$\begin{aligned} p_{n+1,k}(t) &= p_{n,k}(t) + t^{2^n} q_{n,k-\alpha_1}(t) \\ q_{n+1,k}(t) &= q_{n,k}(t) + t^{2^n} p_{n,k-\beta_1}(t) \end{aligned} \tag{5}$$

with initial conditions

$$p_{0,k}(t) = q_{0,k}(t) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Define functions $P_n(s,t)$ and $Q_n(s,t)$ by

$$\begin{aligned} P_n(s,t) &= \sum_{k\in\mathbb{Z}} p_{n,k}(t) e^{iks} \\ Q_n(s,t) &= \sum_{k\in\mathbb{Z}} q_{n,k}(t) e^{iks} \end{aligned}$$

Multiplying the linear recursions (5) by e^{iks} and summing over all integers k, we obtain the following recursions for P_n and Q_n:

$$\begin{aligned} P_{n+1}(s,t) &= P_n(s,t) + t^{2^n} e^{i\alpha_1 s} Q_n(s,t) \\ Q_{n+1}(s,t) &= Q_n(s,t) + t^{2^n} e^{i\beta_1 s} P_n(s,t) \end{aligned}$$

with initial conditions

$$P_0(s,t) = Q_0(s,t) = 1$$

Writing these in matrix form, we have

$$\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & t^{2^n} e^{i\alpha_1 s} \\ t^{2^n} e^{i\beta_1 s} & 1 \end{pmatrix} \begin{pmatrix} P_n \\ Q_n \end{pmatrix}$$

Denoting the two-by-two matrix on the right by A_n, and taking into account the initial conditions, we have

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \left(\prod_{k=0}^{n-1} A_k \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now in this particular case it turns out that the matrices A_k are simultaneously diagonalizable in k, with $B^{-1} A_k B = D_k$, where

$$D_k = \operatorname{diag}\left(1 + t^{2^k} e^{i(\alpha_1+\beta_1)s/2}, 1 - t^{2^k} e^{i(\alpha_1+\beta_1)s/2}\right)$$

and

$$B = \begin{pmatrix} e^{i(\alpha_1+\beta_1)s} & -e^{i(\alpha_1+\beta_1)s} \\ 1 & 1 \end{pmatrix}$$

It follows that our solution can be written

$$P_n(s,t) = \frac{1}{2}\left(\Pi_{n,+} + \Pi_{n,-}\right) + \frac{1}{2} e^{i(\alpha_1-\beta_1)s/2} \left(\Pi_{n,+} - \Pi_{n,-}\right) \tag{6}$$

$$Q_n(s,t) = \frac{1}{2}\left(\Pi_{n,+} + \Pi_{n,-}\right) + \frac{1}{2} e^{-i(\alpha_1-\beta_1)s/2} \left(\Pi_{n,+} - \Pi_{n,-}\right) \tag{7}$$

where

$$\Pi_{n,+} = \Pi_{n,+}(s/2,t) = \prod_{k=0}^{n-1}(1 + t^{2^k} e^{i(\alpha_1+\beta_1)s/2})$$

and

$$\Pi_{n,-} = \Pi_{n,-}(s/2,t) = \prod_{k=0}^{n-1}(1 - t^{2^k} e^{i(\alpha_1+\beta_1)s/2})$$

Recall now that our scheme is progressive. This implies that the polynomials $p_{n,k}$ and $q_{n,k}$ are progressive in an analogous sense:

$$p_{n+1,k} = p_{n,k} + \text{higher order terms}, \qquad q_{n+1,k} = q_{n,k} + \text{higher order terms}$$

In particular, $\lim_{n\to\infty} p_{n,k}(t)$ and $\lim_{n\to\infty} q_{n,k}(t)$ both exist formally for all k, and in fact converge at least for $|t| < 1$. We will denote these limits by $p_k(t)$ and $q_k(t)$ respectively. Similarly P_n and Q_n both have limits as n approaches infinity, which we denote by P and Q. Clearly p_k and q_k are the coefficients of e^{iks} in P and Q, and encode in their exponents the positions in which value k occurs in the limiting sequences $\alpha^\infty = \lim_{n\to\infty} \alpha^{(n)}$ and $\beta^\infty = \lim_{n\to\infty} \beta^{(n)}$. Taking limits in equations (6) and (7) we get

$$\begin{aligned} P(s,t) &= \frac{1}{2}(\Pi_+ + \Pi_-) + \frac{1}{2} e^{i(\alpha_1-\beta_1)s/2}(\Pi_+ - \Pi_-) \\ Q(s,t) &= \frac{1}{2}(\Pi_+ + \Pi_-) + \frac{1}{2} e^{-i(\alpha_1-\beta_1)s/2}(\Pi_+ - \Pi_-) \end{aligned} \tag{8}$$

where

$$\Pi_+ = \Pi_+(s/2,t) = \prod_{k=0}^{\infty}(1 + t^{2^k} e^{i(\alpha_1+\beta_1)s/2})$$

and

$$\Pi_- = \Pi_-(s/2,t) = \prod_{k=0}^{\infty}(1 - t^{2^k} e^{i(\alpha_1+\beta_1)s/2})$$

Expanding $\Pi_+(s,t) = \prod_{k=0}^{\infty}(1 + t^{2^k} e^{i(\alpha_1+\beta_1)s})$ we easily see that

$$\Pi_+(s,t) = \sum_{j=0}^{\infty} \rho_j(t) e^{ij(\alpha_1+\beta_1)s},$$

where

$$\rho_j(t) = \sum_{|\ell|_2 = j} t^\ell$$

Here, $|\ell|_2$ is the number of ones in the binary expansion of ℓ. Thus the sum is over all ℓ whose binary expansions have exactly j ones. Similarly one finds that

$$\begin{aligned} \Pi_-(s,t) &= \prod_{k=0}^{\infty}(1 - t^{2^k} e^{i(\alpha_1+\beta_1)s}) \\ &= \sum_{j=0}^{\infty} (-1)^j \rho_j(t) e^{ij(\alpha_1+\beta_1)s}, \end{aligned}$$

It follows that

$$\frac{1}{2}\left(\Pi_+(s,t)+\Pi_-(s,t)\right)=\sum_{j=0}^{\infty}\rho_{2j}(t)e^{ij(\alpha_1+\beta_1)s}$$

and

$$\frac{1}{2}\left(\Pi_+(s,t)-\Pi_-(s,t)\right)=\sum_{j=0}^{\infty}\rho_{2j+1}(t)e^{ij(\alpha_1+\beta_1)s}$$

Using these in (8) finally yields

$$P(s,t)=\sum_{j=0}^{\infty}\rho_{2j}(t)e^{ij(\alpha_1+\beta_1)s}+\sum_{j=0}^{\infty}\rho_{2j+1}(t)e^{i[j(\alpha_1+\beta_1)+\alpha_1]s}$$
$$Q(s,t)=\sum_{j=0}^{\infty}\rho_{2j}(t)e^{ij(\alpha_1+\beta_1)s}+\sum_{j=0}^{\infty}\rho_{2j+1}(t)e^{i[j(\alpha_1+\beta_1)+\beta_1]s}$$

and the result follows.

Proof of Theorem 4. First we show that composition of pitch patterns is associative. Let α, β, and γ be pitch patterns of length m, n, and p respectively. Calculating the i^{th} entry of $\alpha(\beta\gamma)$, one finds that

$$\left(\alpha(\beta\gamma)\right)_i=\alpha_{q_1}+\beta_{q_2}+\gamma_{r_2} \tag{9}$$

where the indices are nonnegative integers satisfying

$$i=q_1np+r_1, \quad 0\le r_1<np \tag{10}$$
$$r_1=q_2p+r_2, \quad 0\le r_2<p \tag{11}$$

Similarly the i^{th} entry of $(\alpha\beta)\gamma$ is

$$\left((\alpha\beta)\gamma\right)_i=\alpha_{q_4}+\beta_{r_4}+\gamma_{r_3} \tag{12}$$

where

$$i=q_3p+r_3, \quad 0\le r_3<p \tag{13}$$
$$q_3=q_4n+r_4, \quad 0\le r_4<n \tag{14}$$

Note first that

$$r_2=(i \bmod np) \bmod p=i \bmod p=r_3$$

so that the indices on γ are equal in equations (9) and (12). Also, q_3 is the number of p's in i, which by (10) equals the q_1n plus the number of p's in r_1, i.e.

$$q_3=q_1n+q_2 \tag{15}$$

We claim that $q_2=r_4$. From (14) and (15) it suffices to show that $q_2<n$. But this follows directly from (10) and (11). Now $q_1=q_4$ follows from (14) and (15), and we have shown that the indices on α, β, and γ are the same in equations (9) and (12). Thus $\alpha(\beta\gamma)=(\alpha\beta)\gamma$, and composition is associative.

Next we claim the two pitch patterns are right-related if and only if one is transposition of the other. For if $\alpha\mathcal{R}\beta$, then for each $s_1 \in S$ there exists $s_2 \in S$ such that $\alpha s_1 = \beta s_2$. Taking $s_1 = (0)$, we see that the length of α must be a multiple of the length of β. By symmetry, the length of β must be a multiple of the length of α. Hence α and β have the same length. Now $\alpha = \beta s_2$ implies that s_2 must be of length one, and is hence a transposition. This shows that $\mathcal{R} \subseteq \mathcal{T}$. The reverse inclusion follows easily from the fact that transpositions commute with every element of S. Hence $\mathcal{R} = \mathcal{T}$. The proofs for $\mathcal{L}$ and $\mathcal{J}$ are entirely similar, and therefore omitted.

The rest of the theorem is straightforward.