Do these problems in the spaces provided. Show your work.

1. (20 points) Let $X_k, k = 1, 2, 3, \ldots$ be independent and identically distributed random variables, with

$$X_k = \begin{cases} 
-2 & \text{with probability } 1/4 \\
0 & \text{with probability } 1/2 \\
2 & \text{with probability } 1/4
\end{cases}$$

Let $\bar{x}_n = (X_1 + X_2 + \cdots + X_n)/n$, and $S_n = n\bar{x}_n$.

(a) Find the mean and variance of $X_k$.

(b) Find the mean and variance of $\bar{x}_n$.

(c) Use the central limit theorem to estimate $P(|S_{200}| \leq 20)$.

\[ a) \quad E(X_k) = (-2)\frac{1}{4} + 0 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 0 = \mu \]

\[ E(X_k^2) = 4 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 2 \]

\[ \text{Var}(X_k) = E(X_k^2) - E(X_k)^2 = 2 = \sigma^2 \]

\[ b) \quad E(\bar{x}_n) = \mu = 0 \]

\[ \text{Var}(\bar{x}_n) = \frac{\sigma^2}{n} = \frac{2}{n} \]

\[ c) \quad E(S_n) = E(n\bar{x}_n) = nE(\bar{x}_n) = 0 \]

\[ \text{Var}(S_n) = \text{Var}(n\bar{x}_n) = n^2 \text{Var}(\bar{x}_n) = 2n \]

By CLT, $Z = \frac{S_n}{\sqrt{2n}}$ is approximately std. normal

So,

\[ P(-20 \leq S_{200} \leq 20) = P\left(\frac{-20}{\sqrt{200}} \leq \frac{S_{200}}{\sqrt{200}} \leq \frac{20}{\sqrt{200}}\right) \]

\[ = P(-1 \leq Z \leq 1) \approx 0.68 \]
2. (20 points) IQ in a certain population has a mean of \( \mu = 100 \) and a standard deviation of \( \sigma = 15 \).

(a) Pick an individual at random from the population, and let \( X \) be the individual's IQ. Give an upper bound for \( P(|X - 100| > 30) \).

(b) Pick a random sample of size \( n = 25 \) from the population, and let \( \bar{x} \) be the average IQ in the sample. Give an upper bound for \( P(|\bar{x} - 100| > 30) \).

\[ a) \text{ By Chebyshev's Inequality} \]

\[ P( |X - 100| > 30 ) \leq \frac{15^2}{30^2} = \frac{1}{4} \]

\[ b) E(\bar{x}) = 100, \ Var(\bar{x}) = \frac{15^2}{25} , \text{ so} \]

\[ \text{StDev}(\bar{x}) = 3. \text{ Applying Chebyshev's Inequality} \]

\[ P( |\bar{x} - 100| > 30 ) \leq \frac{3^2}{30^2} = \frac{1}{100} \]
3. (20 points) Suppose that the random variable $X$ takes values $[-1, 1]$ and has probability density function

$$p(x) = \frac{3}{2} x^2, \quad x \in [-1, 1].$$

Let $Y = 1 - X^2$. Give the support set for $Y$, and find the probability density function for $Y$.

For $\alpha \in [0, 1]$

$$P(Y \leq \alpha) = P(X \in [-1, -\sqrt{1-\alpha}] \cup [\sqrt{1-\alpha}, 1])$$

$$= \int_{-1}^{-\sqrt{1-\alpha}} \frac{3}{2} x^2 \, dx + \int_{\sqrt{1-\alpha}}^{1} \frac{3}{2} x^2 \, dx$$

$$= 2 \int_{\sqrt{1-\alpha}}^{1} \frac{3}{2} x^2 \, dx$$

$$= x^3 \bigg|_{\sqrt{1-\alpha}}^{1} = \left( 1 - (1-\alpha)^{3/2} \right)$$

So the PDF for $Y$ is

$$p_Y(\alpha) = \frac{d}{d\alpha} P(Y \leq \alpha) = \left( \frac{3}{2} (1-\alpha)^{1/2} \right) = \frac{3}{2} (1-\alpha)^{1/2}$$

for $\alpha \in [0, 1]$. 


4. (20 points) Let \((X,Y)\) be uniformly distributed on the unit disc \(S_{X,Y} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.

(a) Consider the chord of the circle which passes through \((X,Y)\) and is perpendicularly bisected by the line passing through the origin and \((X,Y)\). Let \(L\) be half the length of this chord, as shown in the figure below. Find the probability density function for \(L\).

\[
L^2 + R^2 = 1 \quad \text{so} \quad L^2 = 1 - R^2.
\]

In HW6 we saw that \(R^2\) is uniformly distributed on \([0,1]\). So

\[
P(L \leq \alpha) = P(L^2 \leq \alpha^2) = P(1 - \alpha^2 \leq R^2) = 1 - (1 - \alpha^2) = \alpha^2
\]

So the PDF for \(L\) is

\[
p(\alpha) = \frac{d}{d\alpha} P(L \leq \alpha) = 2\alpha \quad \text{for } \alpha \in [0,1]
\]

(b) Now suppose we have two points \((X_1,Y_1)\) and \((X_2,Y_2)\) chosen independently and uniformly from the unit disc. Let \(D\) be the distance between the two points. Find \(E[D^2]\). (HINT: We saw in Homework 6 that the marginal PDF for \(X_1\) is \(p(x) = 2/\pi \sqrt{1 - x^2}\) for \(x \in [-1,1]\). By symmetry, the marginal PDFs for \(X_2, Y_1\) and \(Y_2\) are the same.)

\[
D^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 = X_1^2 - 2X_1X_2 + X_2^2 + Y_1^2 - 2Y_1Y_2 + Y_2^2
\]

Note:

\[
E(X_1) = \frac{2}{\pi} \int_{-1}^{1} x \sqrt{1 - x^2} \, dx = 0 = E(X_2) = E(Y_1) = E(Y_2).
\]

So by independence

\[
E(X_1X_2) = E(X_1)E(X_2) = 0 \quad \text{Similarly } E(Y_1Y_2) = 0
\]

So

\[
E(D^2) = 4E(X_1^2)
\]

(over)
Now
\[ E(X^2) = \frac{2}{\pi} \int_{-1}^{1} x^2 \sqrt{1-x^2} \, dx = \frac{4}{\pi} \int_0^{\pi/2} x^2 \sqrt{1-x^2} \, dx \]

Let \( x = \sin \theta \), so that \( dx = \cos \theta \, d\theta \). Thus
\[ \int_0^{\pi/2} x^2 \sqrt{1-x^2} \, dx = \int_0^{\pi/2} \sin^2 \theta \cdot \cos^2 \theta \, d\theta \]
\[ = \frac{1}{4} \int_0^{\pi/2} \sin^2(2\theta) \, d\theta \]
\[ = \frac{1}{8} \int_0^{\pi/2} 1 - \cos(4\theta) \, d\theta \]
\[ = \frac{1}{8} \left( \frac{\pi}{2} - \frac{1}{4} \sin 4\theta \right) \bigg|_0^{\pi/2} \]
\[ = \frac{\pi}{16} \]

Thus
\[ E(D^2) = 4E(X^2) \]
\[ = 4 \cdot \frac{4}{\pi} \cdot \frac{\pi}{16} \]
\[ = 1 \]
5. (20 points) Let $X$ and $Y$ be independent and exponentially distributed, both with parameter $\lambda = 1$. Let

$$U = 2X + Y, \quad V = Y$$

Find the support set for $(U, V)$, and the joint probability density function for $(U, V)$.

$$X = \frac{1}{2}(U-V), \quad Y = V$$

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$$S_{U,V} = \{ (u,v) \in \mathbb{R}^2 : u \geq 0, 0 \leq v \leq u \}$$

$$f_{X,Y}(x,y) = e^{-(x+y)}, \quad x, y \geq 0$$

So the PDF for $(U, V)$ is

$$f_{U,V}(u,v) = \frac{1}{2} \exp \left(-\frac{1}{2}(u-v) - v\right) = \frac{1}{2} e^{-\frac{1}{2}(u+v)} \quad \text{for} \quad (u,v) \in S_{U,V}$$
**Bonus**: (10 points) In a class of 100 students, the mean score on the first exam is 75, with a standard deviation of 6. What is the maximum number of students who could have made 100 on the exam?

Pick a student randomly and uniformly from the class, and let $X$ be the selected student's score. Then for all $\varepsilon \in (0, 1)$ we have

$$\frac{\# \text{ making } 100\%}{100} = P(X = 100)$$

$$\leq P(X > 99 + \varepsilon)$$

$$\leq P(1|X - 75| > 24 + \varepsilon)$$

$$= \frac{6^2}{(24 + \varepsilon)^2} \quad (\forall \varepsilon \in (0, 1))$$

Therefore

$$\frac{\# \text{ making } 100\%}{100} \leq \lim_{\varepsilon \to 1} \frac{6^2}{(24 + \varepsilon)^2} = \frac{6^2}{25^2}$$

So

$$\frac{\# \text{ making } 100\%}{100} \leq \frac{6^2}{25^2} \cdot 100 = 5.76$$

So no more than 5 students (all) have made 100%. 